ON GROUP-VELOCITY

By Horace Lamb, F.R.S.

By a "group" is meant, in this connection, a long succession of waves in which the distance between successive crests, and the amplitude, vary very slightly. The well-known formula connecting the group-velocity U with the wave-velocity V, viz.,

$$U = V - \lambda \frac{dV}{d\lambda}, \tag{1}$$

was obtained originally by considering the result of superposing two infinite simple harmonic wave-trains of equal amplitude and slightly different frequency.[*] The extension to a more general type of group was made by Rayleigh[†] and Gouy.[‡] The argument of these writers admits of being put very concisely. Assuming a disturbance

$$y = \Sigma C \cos(nt - kx + \epsilon), \tag{2}$$

where the summation (which may of course be replaced by an integration) embraces a series of terms in which the values of n, and therefore also of k, vary very slightly, we remark that the phase of the typical term at time $t + \Delta t$ and place $x + \Delta x$ differs from the phase at time t and place x by the amount $n\Delta t - k\Delta x$. Hence if the variations of n and k from term to term be denoted by δn and δk, the change of phase will be sensibly the same for all the terms, provided

$$\delta n \Delta t - \delta k \Delta x = 0. \tag{3}$$

The group as a whole therefore travels with the velocity

$$U = \frac{\Delta x}{\Delta t} = \frac{dn}{dk}. \tag{4}$$

Since $n = kV$, $k = 2\pi/\lambda$, this agrees with (1).

[*] Rayleigh, *Proc. London Math. Soc.*, Vol. IX., p. 21 (1877); *Theory of Sound*, § 191.

[†] *Nature*, Vol. XXV., p. 52 (1881); *Scientific Papers*, Vol. I., p. 540.

[‡] "Sur la vitesse de la lumière," *Ann. de Chim. et de Phys.*, Vol. XVI. p. 262 (1889).

Another derivation of (1) can be given which is, perhaps, more intuitive. In a medium such as we are considering, where the wave-velocity varies with the frequency, a limited initial disturbance gives rise in general to a wave-system in which the different wave-lengths, travelling with different velocities, are gradually sorted out.* If we regard the wave-length λ as a function of x and t, we have

$$\frac{\partial \lambda}{\partial t} + U\frac{\partial \lambda}{\partial x} = 0, \qquad (5)$$

since λ does not vary in the neighbourhood of a geometrical point travelling with velocity U; this is, in fact, the definition of U. Again, if we imagine another geometrical point to travel with the *waves*, we have

$$\frac{\partial \lambda}{\partial t} + V\frac{\partial \lambda}{\partial x} = \lambda\,\frac{\partial V}{\partial x} = \lambda\,\frac{dV}{d\lambda}\,\frac{\partial \lambda}{\partial x}, \qquad (6)$$

the second member expressing the rate at which two consecutive wave-crests are separating from one another. Combining (5) and (6), we are led, again, to the formula (1).

It has been pointed out by the writer† that this formula admits of a simple geometrical representation : viz., if a curve be constructed with λ as abscissa and V as ordinate, the value of U for any assigned value of λ is given by the intercept which the corresponding tangent to the curve makes on the axis of ordinates.

A question of some interest arises, as to whether it is possible for the group-velocity, in any real case, to be negative.‡ This was proposed to the writer by Prof. Schuster, who pointed out that the optical formulæ relating to anomalous dispersion indicate a negative group-velocity for certain portions of the spectrum lying within the regions of special absorption. This appears readily from the curves§ by which the refractive index (μ) is exhibited as a function of the wave-length *in vacuo* (λ_0). The formula

$$\frac{V_0}{U} = \mu - \lambda_0\,\frac{d\mu}{d\lambda_0}, \qquad (7)\|$$

which is easily derived from (1), shows that the group-velocity is now *in-*

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* As, for example, in the waves on deep water due to a local disturbance of finite duration (Cauchy, Poisson).

† See *Manchester Mem.*, Vol. XLIV., No. 6 (1900), where a number of examples are given.

‡ That is, to have the opposite sign to the wave-velocity.

§ See, for example, Drude, *Lehrbuch d. Optik*, p. 362.

‖ Gouy, *loc. cit.*

versely proportional to the intercept made by the tangent on the axis of ordinates, and it is evident on inspection that for certain ranges of λ_0 this intercept is, in the case referred to, negative. It is to be observed, however, that the notion of group-velocity, as hitherto developed, ceases to be applicable when the absorption is so intense that the distance within which the amplitude falls to (say) one half of its original value becomes comparable with the wave-length.

It is easy, however, to devise mechanical arrangements, free from dissipation, in which the group-velocity shall be negative. We may begin, for instance, with a long straight wire subject to a longitudinal thrust (P), instead of a tension. This is of course unstable, but we may obtain stability for waves exceeding a certain length, if we imagine, in addition, that every point of the wire is attracted towards its equilibrium position by a force varying as the distance. The equation of transverse vibration is now of the form

$$\frac{\partial^2 y}{\partial t^2} + p^2 y + c^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{8}$$

where $c^2 = P/\rho$, ρ being the density. Assuming a solution

$$y = C \cos{(nt - kx)}, \tag{9}$$

we find $\qquad k^2 = (p^2 - n^2)/c^2, \qquad UV = -c^2, \tag{10}$*

provided $k < p/c$.

We may secure stability for all values of λ by further endowing the wire with a sufficient degree of stiffness. The equation of motion then assumes the form†

$$\frac{\partial^2 y}{\partial t^2} + p^2 y + c^2 \frac{\partial^2 y}{\partial x^2} + a^2 \kappa^2 \frac{\partial^4 y}{\partial x^4} = 0, \tag{11}$$

which gives $\quad n^2 = p^2 - k^2 c^2 + a^2 \kappa^2 k^4, \quad UV = -c^2 + 2\kappa^2 a^2 k^2. \tag{12}$

Hence n will be real for all values of k provided $c^2 < 2\kappa a p$, whilst U will have the opposite sign to V for waves so long that $k^2 < \frac{1}{2} c^2/\kappa^2 a^2$. It may be noticed that vibrations below a certain frequency (determined by $n^2 = p^2 - \frac{1}{4} c^4/\kappa^2 a^2$) cannot be propagated. For sufficiently long waves

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* The curve exhibiting the relation between V and λ has the form of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

† See Rayleigh, *Sound*, § 188.

the flexural term in (11) is insensible, and we fall back on the former case.

Another example, in which there may be negative group-velocity for all wave-lengths, together with thorough stability, is furnished by an arrangement employed by Lord Rayleigh* in a different connection. This consists of a tense wire attracted to its equilibrium position by a force varying as the distance, and endowed with rotatory inertia, but of negligible stiffness.† The differential equation is now of the form

$$\frac{\partial^2 y}{\partial t^2} + p^2 y - a^2 \frac{\partial^2 y}{\partial x^2} - \kappa^2 \frac{\partial^4 y}{\partial x^2 \partial t^2} = 0, \tag{13}$$

where $a^2 = T/\rho$, T being the tension, and κ is a radius of gyration. This gives

$$n^2 = \frac{p^2 + k^2 a^2}{1 + \kappa^2 k^2}, \qquad k^2 = \frac{p^2 - n^2}{\kappa^2 n^2 - a^2}, \tag{14}$$

and thence

$$UV = -\frac{\kappa^2 p^2 - a^2}{(1 + \kappa^2 k^2)^2} = -\frac{(\kappa^2 n^2 - a^2)^2}{\kappa^2 p^2 - a^2}. \tag{15}$$

The condition for negative group-velocity is therefore $\kappa^2 p^2 > a^2$. It will be observed that vibrations for which $a/\kappa < n < p$ can alone be propagated.

It is known that the velocity U, determined by (1), also expresses the rate at which energy is propagated.‡ The proof applies without qualification to such cases as we have been considering; and we infer, both from this and from the former line of argument, that when a local disturbance is produced in a medium having the property in question groups of waves will be propagated outwards, whilst the individual waves composing a group will be found to be moving inwards. It may be worth while to illustrate this by means of one of our mechanical arrangements; for simplicity we may take the one first considered. It will help to make the matter clearer if we introduce a slight degree of viscosity, so that the equation (8) is replaced by

$$\frac{\partial^2 y}{\partial t^2} + p^2 y + \nu \frac{\partial y}{\partial t} + c^2 \frac{\partial^2 y}{\partial x^2} = 0. \tag{16}$$

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* *Phil. Mag.*, Vol. xlvi. (1898) ; *Scientific Papers*, Vol. iv., p. 369.

† This might be realized by a cylindrical wire with a series of close equidistant peripheral cuts extending nearly to the axis.

‡ Cf. Osborne Reynolds, *Nature*, Vol. xvi., p. 343 (1877), *Scientific Papers*, Vol. i., p. 198, for the case of water waves. The proof for the general case was given by Rayleigh in the paper cited first on p. 473 *ante*.

If we suppose that a forced vibration

$$y = C \cos nt \tag{17}$$

is maintained at the origin, the disturbance established to the right of O will be found by assuming

$$y = C e^{int + ax}, \tag{18}$$

provided that in the end we retain only the real part. Substituting in (16), we have

$$a^2 = -(p^2 - n^2 + i\nu)/c^2, \tag{19}$$

and therefore, on account of the assumed smallness of ν,

$$a = \pm ik \mp \nu/2kc^2, \tag{20}$$

where k is the positive quantity determined by (10). In order to secure finiteness for $x = \infty$, the *upper* signs must be taken, and the realized solution is

$$y = C e^{-\nu x/2kc^2} \cos (nt + kx), \tag{21}$$

representing a train in which the individual waves are travelling *inwards* towards the source of disturbance.

If we regard ν as infinitesimal, the mean energy per unit length will be double the mean kinetic energy,* and is therefore equal to $\frac{1}{2}\rho n^2 C^2$. On the other hand, the rate at which work is supplied at the origin is

$$P \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} = PknC^2 \sin^2 (nt + kx), \tag{22}$$

the mean value of which is $\frac{1}{2}PknC^2$. This is equal to the energy in a length U, if

$$U = \frac{k}{n} \frac{P}{\rho} = \frac{k}{n} c^2. \tag{23}$$

This agrees with (10), since V is now equal to $-n/k$.

We may also illustrate the problem of reflection and transmission at a point where the properties of the medium are discontinuous. Referring to Lord Rayleigh's modification of the tense wire, we may suppose that to the right of the origin the equation (13) holds, whilst to the left of O

$$\frac{\partial^2 y}{\partial t^2} - a_0^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{24}$$

where $a_0^2 = T/\rho_0$. Taking the amplitude of the waves incident from the

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* This familiar relation ceases to hold when ν has to be taken into account.

left as unity, we assume

$$y = e^{i\,(nt-k_0 x)} + A\,e^{i(nt+k_0 x)} \quad [x < 0], \tag{25}$$

$$y = B\,e^{i(nt+kx)} \quad\quad\quad [x > 0], \tag{26}†$$

so that A denotes the amplitude of the reflected, B of the transmitted, waves. The conditions to be satisfied at the junction $x = 0$ are to be found from the consideration that to the right of O we have a shearing force

$$F = \kappa^2 \rho\,\frac{\partial^3 y}{\partial x\,\partial t^2}, \tag{27}$$

whilst to the left of O we have $F = 0$. Since the values of y and $F + T\,\partial y/\partial x$ must evidently be continuous, we have

$$1 + A = B, \tag{28}$$

$$k_0 \rho_0 a_0^2 (1-A) = k\rho\,(\kappa^2 n^2 - a^2)\,B, \tag{29}$$

whence we find

$$B = 2 \Big/ \left\{ 1 + \frac{k}{k_0} \left(\frac{\kappa^2 n^2}{a^2} - 1 \right) \right\} \tag{30}$$

This may be expressed in terms of n and constant quantities by means of (14) and the relation $k_0^2 = n^2/a_0^2$; thus

$$B = 2 \Big/ \left\{ 1 + \frac{a_0}{a} \sqrt{\left(\frac{p^2}{n^2} - 1 \right)} \sqrt{\left(\frac{\kappa^2 n^2}{a^2} - 1 \right)} \right\}. \tag{31}$$

If we take the real parts of the expressions in (25) and (26), then, corresponding to an incident vibration

$$y = \cos\,(nt - k_0 x), \tag{32}$$

we have a transmitted vibration

$$y = B \cos\,(nt + kx). \tag{33}$$

The potential energy per unit length at any point to the right of O is

$$\tfrac{1}{2} \left\{ \rho p^2 y^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right\},$$

the mean value of which is $\tfrac{1}{4}\rho\,(p^2 + k^2 a^2)\,B^2$. The mean total energy per unit length will be twice this. Hence, if I denote the ratio of the

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† The necessity for the adoption of $nt + kx$ instead of the usual $nt - kx$ in the exponential might be made clearer by the introduction of a small frictional term into the equation (13). It would then appear that an assumption of the form $y = B\,e^{i\,(nt - kx)}$ would involve the existence of a source at $x = +\infty$.

transmitted to the incident energy, per unit time, we shall have

$$I = \frac{\frac{1}{2}\rho\,(p^2+k^2a^2)\,U}{\frac{1}{2}\rho_0\,n^2a_0}\,B^2. \tag{34}$$

By means of (14) and (15) this can be put in a variety of forms : thus for example

$$I = \frac{k}{k_0}\left(\frac{\kappa^2n^2}{a^2}-1\right)B^2. \tag{35}$$

Comparing this with (30), we find

$$I = (2-B)\,B. \tag{36}$$

This is in accordance with the principle of energy, for, if I' denote the ratio of the energy of the reflected to that of the incident waves, it makes

$$I+I' = (2-B)\,B+A^2 = 1, \tag{37}$$

by (28).

It is hardly to be expected that the notion of a negative group-velocity will have any very important physical application; but the preceding discussion may serve to emphasize the point that ideas of wave-propagation acquired in the study of air-waves (for example) need to be used with some caution when we are dealing with a dispersive medium.