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RUNAWAY OF LINE-DRIVEN WINDS TOWARD CRITICAL AND OVERLOADED SOLUTIONS

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ABSTRACT

Line-driven winds from hot stars and accretion disks are thought to follow a unique, critical solution that corresponds to a maximum mass-loss rate and a particular velocity law. We show that in the presence of negative velocity gradients, radiative-acoustic (Abbott) waves can drive shallow wind solutions toward larger velocities and mass-loss rates. Perturbations that are introduced downstream from the critical point of the wind lead to a convergence toward the critical solution. By contrast, low-lying perturbations cause evolution toward a mass-overloaded solution, developing a broad deceleration region in the wind. Such a wind differs fundamentally from the critical solution. For sufficiently deep-seated perturbations, overloaded solutions become time-dependent and develop shocks and shells.

Subject headings: accretion, accretion disks — galaxies: active — hydrodynamics — novae, cataclysmic variables — stars: mass loss — stars: winds, outflows

1. INTRODUCTION

Atmospheres of hot luminous stars and accretion disks in active galactic nuclei and in cataclysmic variables form extensive outflows because of the super-Eddington radiation fluxes in UV resonance and subordinate lines. An understanding of these winds is hampered by the pathological dependence of the driving force on the flow velocity gradient. Castor, Abbott, & Klein (1975, hereafter CAK) found that line-driven winds (LDWs) from O stars should follow a unique, critical state that corresponds to a maximum mass-loss rate. The equation of motion for a one-dimensional, spherically symmetric, polytropic outflow that is subject to a Sobolev line force allows for two infinite families of so-called shallow and steep solutions. However, none of these families can provide for a *global* solution alone. Shallow solutions do not reach infinity, while steep solutions do not extend into the subsonic regime, including the photosphere. The critical wind starts then as the fastest shallow solution and switches at the critical point in a continuous and differentiable manner to the slowest steep solution. Hence, the critical point and not the sonic point determines the bottleneck in the wind. This description, in principle, applies equally to winds from stars and accretion disks.

A physical interpretation of the CAK critical point was given by Abbott (1980), who derived a new type of radiative-acoustic wave (hereafter Abbott waves). These waves can propagate inward, in the stellar rest frame, only from below the CAK critical point. Above the critical point, they are advected outward. Hence, the CAK critical point serves as an information barrier, much like the sonic or Alfvén points in thermal and hydromagnetic winds. Abbott's analysis was challenged by Owocki & Rybicki (1986), who found that for a pure absorption LDW, the signal speed is the sound speed and not the much faster Abbott speed. As noted already by these authors, this should be the consequence of assuming pure line absorption, which does not allow for any radiatively modified, inward-wave mode. Meanwhile, there is ample evidence for Abbott waves in time-dependent wind simulations (Owocki & Puls 1999).

Shallow solutions fail to reach infinity because they cannot perform the required spherical expansion work, implying that the flow starts to decelerate. Since this usually occurs very far out in the wind, the local wind speed is much larger than the local escape speed, and the wind escapes to infinity. Thus, a simple generalization of the CAK model, allowing for flow deceleration, renders shallow solutions globally admissible. This raises a fundamental question of why the wind would adopt the critical solution at all, and attain the critical mass-loss rate and velocity law, as proposed by CAK.

In this Letter, we analyze the physical mechanism that drives shallow solutions toward the critical solution, and we discuss under what conditions this evolution does not terminate at the CAK solution but continues into the realm of overloaded solutions. We find that, so far, simulations were affected by the numerical runaway toward the critical solution, by not accounting for Abbott waves in the Courant time step.

2. ABBOTT WAVES

Abbott waves are readily derived by bringing the wind equations into characteristic form. We consider a one-dimensional planar wind of velocity $v(z, t)$ and density $\rho(z, t)$, assuming zero sound speed. The continuity and Euler equations are, respectively, as follows:

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial z} + \rho\frac{\partial v}{\partial z} = 0, \tag{1}$$

$$E \equiv \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + g(z) - CF(z)\left(\frac{\partial v/\partial z}{\rho}\right)^{\alpha} = 0. \tag{2}$$

Here $g(z)$ and $F(z)$ are the gravity and the radiative flux, respectively. The CAK line force is given by $g_{\rm l} \equiv CF(z)(v'/\rho)^{\alpha}$ (with $v' \equiv \partial v/\partial z$), with constant C and exponent $0 < \alpha < 1$. The unique stationary CAK wind, $v_{\rm cr}(z)$, $\rho_{\rm cr}(z)$, is found by requiring a critical point at some $z_{\rm cr}$. The number of solutions for $vv'(z)$ changes from 2 to 1 at $z_{\rm cr}$ (which is a saddle point); hence, $\partial E/\partial(vv')_{\rm cr} = 0$ holds. Writing C in terms of critical point

quantities, the Euler equation becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + g(z) - \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{F(z)}{F(z_{\rm cr})} \times g(z_{\rm cr})^{1-\alpha}(\rho_{\rm cr}v_{\rm cr})^{\alpha}\left(\frac{\partial v/\partial z}{\rho}\right)^{\alpha} = 0. \quad (3)$$

Note that for stationary planar winds, ρv is constant. If, in addition, g and F are taken to be constant with height and if $\rho_{\rm cr}v_{\rm cr}v'/\rho$ is replaced by $vv'/\dot{m}$, with a normalized mass-loss rate $\dot{m} \equiv \rho v/\rho_{\rm cr}v_{\rm cr}$, one finds that E no longer depends explicitly on z for stationary solutions. Hence, vv' is independent of z, too. This implies that $z_{\rm cr}$ is ill-defined and that every point of the CAK solution is a critical point. CAK removed this degeneracy by introducing gas pressure terms. Here we take a different approach and assume $g = z/(1 + z^2)$. A situation with roughly constant radiative flux and gravity showing a maximum at finite height could be encountered above isothermal disks around compact objects (see Feldmeier & Shlosman 1999). The critical point is determined by the regularity condition, $dE/dz_{\rm cr} = 0$; hence, $z_{\rm cr} = 1$, and the critical point coincides with the gravity maximum. For simplicity, we also choose $\alpha = \frac{1}{2}$ from now on, which is reasonably close to realistic values of $\alpha \leq \frac{2}{3}$ (Puls, Springmann, & Lennon 2000). None of our results should depend qualitatively on the assumptions made so far. The Euler equation is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + g(z) - 2\sqrt{g_{\rm cr}\rho_{\rm cr}v_{\rm cr}}\sqrt{\frac{\partial v/\partial z}{\rho}} = 0, \quad (4)$$

where $g_{\rm cr} \equiv g(z_{\rm cr})$. The stationary solutions for wind acceleration are given by

$$vv'(z) = \frac{g_{\rm cr}}{\dot{m}}\left[1 \pm \sqrt{1 - \frac{\dot{m}g(z)}{g_{\rm cr}}}\right]^2, \quad (5)$$

where the plus and minus signs refer to steep and shallow solutions, respectively. For $\dot{m} \leq 1$, shallow and steep solutions are globally (i.e., everywhere) defined. For $\dot{m} > 1$, solutions are called *overloaded*, and they become imaginary in the neighborhood of the gravity maximum. These winds carry mass-loss rates that are too large, and they eventually stagnate.

Next we put the Euler equation into quasi-linear form, which does *not* mean that we linearize it. Differentiating E with respect to z (Courant & Hilbert 1962; Abbott 1980) and introducing $f \equiv \partial v/\partial z$, equations (1) and (4) become, respectively,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)\rho + \rho f = 0, \quad (6)$$

$$\left[\frac{\partial}{\partial t} + (v + v_{\rm A})\frac{\partial}{\partial z}\right]\frac{f}{\rho} + \frac{1}{\rho}\frac{\partial g}{\partial z} = 0, \quad (7)$$

with the inward Abbott speed in the rest frame, $v_{\rm A} \equiv -(g_{\rm cr}v/\dot{m}v')^{1/2}$. In the WKB approximation, individual spatial and temporal variations are much larger than the inhomogeneous term g'/ρ in equation (7), and so the term can be neglected. Consequently, v'/ρ is a Riemann invariant propagating at characteristic speed $v + v_{\rm A}$. Perturbations of v'/ρ correspond to the amplitude of a wave propagating at phase speed $v + v_{\rm A}$. Note that v'/ρ is proportional to the Sobolev line optical depth, indicating that this wave is a true radiative mode.

The second characteristic is determined by the continuity equation (6). In the advection operator in parentheses, v has to be read as $v + 0$ in the zero sound speed limit. This outward-propagating invariant corresponds to a sound wave, with amplitude ρ scaling with gas pressure.

At the critical point, $\dot{m} = 1$ and $vv'(z_{\rm cr}) = g(z_{\rm cr})$ after equation (5); hence, $v_{\rm A,\,cr} = -v_{\rm cr}$ [where we introduced $v_{\rm A,\,cr} \equiv v_{\rm A}(z_{\rm cr})$]. Abbott waves stagnate at the critical point, analogous to the sound waves at the sonic point. For shallow solutions, $\dot{m} < 1$ and $vv' < v_{\rm cr}v'_{\rm cr}$ from equation (5); hence, $v + v_{\rm A} < 0$. Shallow LDW solutions are therefore the subcritical analog to solar wind breezes.

Because, in the rest frame, the inward Abbott mode can propagate at larger absolute speeds than the outward sound mode, Abbott waves can determine the Courant time step in time-explicit hydrodynamic simulations. Violating the Courant step results in numerical instability. Despite this fact, Abbott waves along shallow solutions were never considered in the literature.

3. WIND CONVERGENCE TOWARD THE CRITICAL SOLUTION

We turn our attention to the physical mechanism that can drive LDWs away from shallow solutions and toward the critical one. Starting from an arbitrary shallow solution as the initial condition, we explicitly introduce perturbations at some fixed location in the wind and study their evolution. In order to keep *unperturbed* shallow solutions stable in numerical simulations, we fix one outer boundary condition, according to inward-propagating Abbott waves. A constant mass-loss rate at either the outer boundary condition or the nonreflecting boundary condition (Hedstrom 1979) serves this aim. At the inner, subcritical boundary, we also fix one boundary condition, according to incoming sound waves. Nonreflecting boundary conditions and ρ = const give similar results.

Wind convergence toward the critical solution is then triggered by negative flow velocity gradients. Allowing for $v' < 0$ turns the inward Abbott mode of phase speed $v + v_{\rm A} < 0$ in the rest frame into an outward-propagating mode. This is readily seen for a line force that is zero for negative v', i.e., when all photons are absorbed at a resonance location between the photosphere and the wind point. The Euler equation is simplified to that for an ordinary gas, with characteristic speed $v - 0 > 0$ in the zero sound speed limit. At the other extreme, for a purely local line force in which the unattenuated stellar or disk radiation field reaches the wind point, $g_{\rm l} \propto (|v'|)^{1/2}$. Here the Abbott phase speed is found to be $v + v_{\rm A}$, with $v_{\rm A} = +(-g_{\rm cr}v/\dot{m}v')^{1/2}$ for $v' < 0$.

Consider then a sawtooth-like velocity perturbation (a sinusoidal perturbation leads to similar results). Slopes $v' > 0$ propagate inward, and slopes $v' < 0$ propagate outward. Hence, as a kinematical consequence, a sawtooth that is initially symmetric with respect to the underlying stationary velocity law evolves toward larger velocities. This is demonstrated in Figure 1, where, in the course of time, a periodic sawtooth perturbation is introduced at $z = 2$. The line force is assumed to be proportional to $(|v'|)^{1/2}$, and the initial shallow solution has $\dot{m} = 0.8$. The figure shows $2\frac{1}{2}$ perturbation cycles. For upward-pointing kinks, the slopes propagate apart, and a flat velocity law develops between them. At each time step dt, a new increment $dv = 4\,dt\,\delta v/T$ (δv and T being the amplitude and period of the sawtooth, respectively) is added at $z = 2$; hence, the flattening velocity law does not show up in region A of Figure 1. Overall, the wind speed at the perturbation site

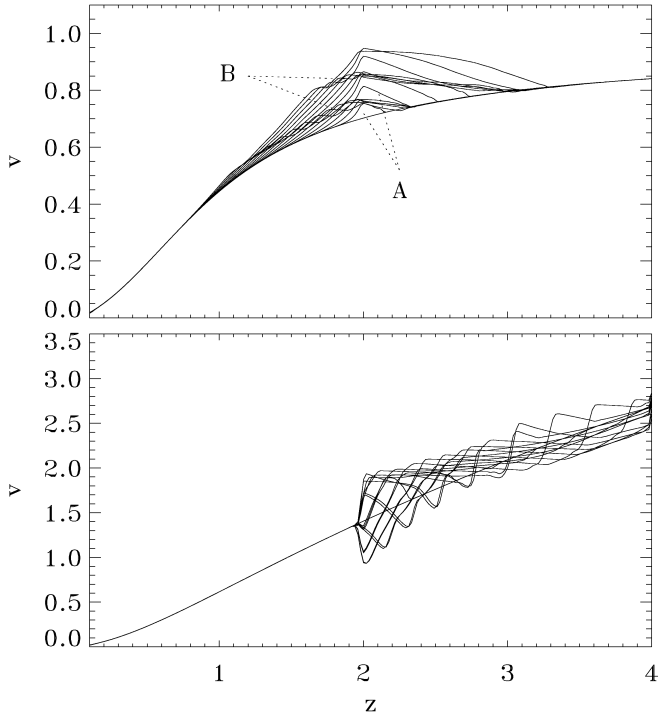

FIG. 1.—*Top*: Evolution of a shallow wind during $2\frac{1}{2}$ periods of a sawtooth perturbation of amplitude, with 10% being introduced at $z = 2$. Regions "A" and "B" correspond to phases in which upward- and downward-pointing kinks, respectively, are introduced into the flow. *Bottom*: Stable Abbott wave excitation in the critical wind, at 35% perturbation amplitude.

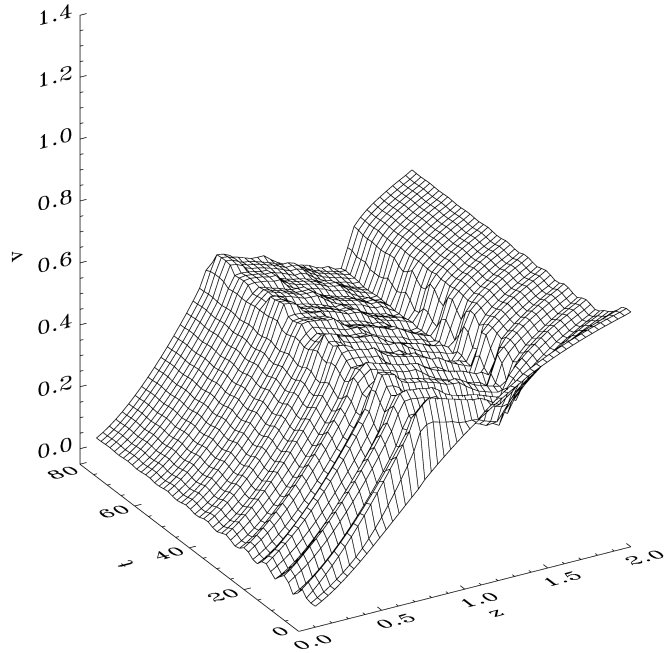

FIG. 2.—Wind evolution toward a stationary, overloaded solution showing an extended decelerating region. A periodic sawtooth perturbation is introduced into a shallow solution at $z = 0.8$, upstream from the critical point at $z_{cr} = 1$.

evolves toward larger values during these phases. On the other hand, for downward-pointing kinks of the sawtooth, $-\delta v$, the two approaching slopes merge, and the wind speed evolves back toward its unperturbed value after each decrement $-dv = -4\, dt\, \delta v/T$. The wind velocity hardly evolves during these phases (see region B of Fig. 1). Over a full perturbation cycle, the wind speed clearly increases.

Essentially, any perturbation that introduces negative v' will accelerate the wind. The amplitude of the perturbation is rather irrelevant since, with decreasing perturbation wavelength, negative v' occur at ever smaller amplitudes. However, in more realistic winds, dissipative effects may smear out short-scale perturbations before they can grow. Details of the physical mechanism will be discussed elsewhere.

If the perturbation lies downstream from the critical point, the wind converges toward the critical solution. Namely, as soon as the perturbation site comes to lie on the supercritical part of the CAK solution during its evolution, positive velocity slopes propagate *outward* and combine with negative slopes to a full wave train. No information is propagated upstream. This unconditional stability of the outer CAK solution is shown in the bottom panel of Figure 1.

4. WIND CONVERGENCE TOWARD OVERLOADED SOLUTIONS

Wind runaway toward larger speeds, as caused by perturbations introduced *upstream* from the critical point, does not terminate at the critical CAK solution. For low-lying perturbations, communication with the wind base is still possible once the subcritical branch of the CAK solution is reached. The wind gets further accelerated into the domain of mass-overloaded solutions (where $vv' > v_{cr}v'_{cr}$, and hence $v > v_{cr}$ for $z < z_{cr}$ according to eq. [5]) until a generalized critical point develops, which prevents the inward propagation of Abbott waves and the adjustment of the mass-loss rate. Such generalized critical points are given by "termination" points z_t of overloaded solutions, where the velocity becomes imaginary. At z_t, the number of real solutions $vv'(r)$ changes from 2 (shallow and steep) to 0. Hence, termination points are defined by the same condition as the CAK critical point (at which the number of solutions changes from 2 via 1 to 2), $\partial E/\partial(vv')_t = 0$. From the stationary version of equation (4), $v_{A,t} = -v_t$; hence, Abbott waves stagnate at termination points, and the latter become generalized critical points.

The fact that perturbations with negative v' accelerate the wind to either a critical or an overloaded state can be cast into the black hole conjecture (Penrose 1965): an LDW avoids a "naked" base by enclosing it with a critical surface.

Since to each z_t there corresponds a unique, supercritical mass-loss rate, the latter is determined by the perturbation *location* alone. Using $v_{A,t} = -v_t$, one finds $\dot{m}_t = g_{cr}/g_t > 1$ for a planar wind with constant radiative flux.

At a termination point, vv' jumps to the decelerating branch, $vv' < 0$. Beyond a well-defined location above the gravity maximum, the super-CAK mass-loss rate can again be lifted by the line force, and vv' jumps back to the accelerating branch. Hence, two stationary kinks occur in the velocity law. Figure 2 shows a hydrodynamic simulation of the evolution toward an overloaded solution. Sawtooth-type velocity perturbations were introduced at $z = 0.8$. Correspondingly, $\dot{m} = 1.025$ for the overloaded solution, using $g = z/(1 + z^2)$.

Future work has to clarify whether or not LDWs show deep-seated perturbations. It seems unlikely, however, that they would occur at a unique location. Hence, overloaded winds should be nonstationary and should show a *range* of supercritical mass-loss rates.

More fundamentally, time-dependent overloaded solutions occur already for single, unique perturbation sites once these sites lie below a certain height. For the present wind model,

this is at $z \approx 0.66$. The overloading is then so severe and the decelerating region so broad that negative wind speeds result (see Poe, Owocki, & Castor 1990). The corresponding mass-loss rates are still only a few percent larger than the CAK value. The gas that falls back toward the photosphere collides with the outflowing gas, and a time-dependent situation develops. Within each perturbation period, a shock forms in the velocity law, supplemented by a dense shell. These shocks and shells propagate outward (A. Feldmeier & I. Shlosman 2000, in preparation).

Although strong perturbations introducing negative velocity gradients can appear already in O star winds, accretion disk winds are the prime suspects. The reasons for this are that accretion processes and their radiation fields in cataclysmic variables and galactic nuclei are intrinsically variable on a range of timescales (Frank, King, & Raine 1992) and that disk LDWs are driven by a combination of uncorrelated, local, and central radiation fluxes.

5. SUMMARY

We find that shallow solutions to line-driven winds are subcritical with respect to Abbott waves (sub-Abbottic). These waves cause shallow solutions to evolve toward larger speeds and mass-loss rates because of the asymmetry of the line force with regard to positive and negative velocity gradients and because perturbations with opposite signs of dv/dz propagate in opposite directions. Steep velocity slopes propagate toward the wind base, steepen the inner wind, and lift it to higher mass-loss rates. In the presence of enduring wind perturbations, this proceeds until a critical point forms and Abbott waves can no longer penetrate inward.

The resulting solution does not necessarily correspond to the CAK wind. For perturbations that originate below the critical point, the developing Abbott wave barrier is found to be the termination point of a mass-overloaded solution. The velocity law acquires a kink at the termination point, where the wind starts to decelerate. Whether the wind converges to a critical or an overloaded solution depends entirely on the *location* of perturbations and not, e.g., on the boundary conditions at the wind base.

If Abbott waves are not accounted for in the Courant time step of hydrodynamic simulations, we find that *numerical* runaway can drive the solution toward the critical CAK wind. A detailed discussion of this will be given elsewhere.

Future work has to clarify whether and where the perturbations causing the local flow deceleration, $dv/dz < 0$, can occur in LDWs. Overloaded winds may be detected observationally. While their mass-loss rates should still be close to CAK values, broad regions of decelerating flow could be identified in P Cygni line profiles. Furthermore, besides the shocks from the line-driven instability (Lucy 1982; Owocki, Castor, & Rybicki 1988), the shocks occurring in overloaded solutions with infalling gas may contribute to the X-ray emission from LDWs. Note that the present wind runaway occurs already in the lowest order Sobolev approximation and is therefore unrelated to the line-driven instability that depends on velocity curvature terms (Feldmeier 1998).

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