weiter Rossby nach Pedlosky 16.3.22

S.105, β-Plane. Grandiose Idee Rossbys, (1939)
lokale Tangent. ebenen an Erde einzuführen
nur mit unterschiedlichem Coriolisparameter f

$$\Pi = \frac{\zeta + f}{H} \qquad \text{pot. vort.}$$

$$df = \frac{1}{R}\,\frac{df}{d\theta}\,\underbrace{R\,d\theta}_{= dy \text{ oder } y} = \beta y \;,\quad \beta = \frac{2\Omega}{R}\cos\theta$$

$$f(y) = f_0 + \beta y \;,\qquad f_0 = 2\Omega \sin\theta$$

$$H = D + \eta - h_B$$

$$\Pi = \frac{f_0 + \beta y + f_0\, h_B/D + \zeta - f_0\,\eta/D}{D}$$

nach Linearisieren

Die Terme/Beiträge βy und $f_0 h_B/D$ sind in Π nicht zu unterscheiden.

daher identische Effekte = Rossbywellen

a) durch Bodenprofil $h_B \cdot f_0/D$

b) durch Variation des Coriolisbeschl. βy

variation of topography ↓ β-Effekt

Quasigeostrophe Potential Vorticity Equation

16.3.22

Vorbemerkung: dies etwas konfus in Pedlosky, über viele Seiten verteilt.

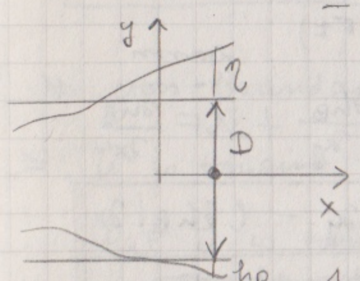
z.B. S. 86-92 (undurchsichtige Skalenanalyse)

Annahmen: — shallow water

die Rossbywelle IST in geostrophe Balance! → (quasi-) geostrophic

— potential vorticity conservation

— Linearization / Skalierung



$$\Pi = \frac{f+\zeta}{h} = \frac{f+\zeta}{D+\eta-h_B} = \frac{f+\zeta}{D\left(1+\frac{\eta}{D}-\frac{h_B}{D}\right)}$$

$$\Pi \approx \overset{3}{\frac{f}{D}} + \overset{1}{\frac{\zeta}{D}} - \overset{2}{\frac{f\eta}{D^2}} + \underset{4}{\frac{f h_B}{D^2}} \qquad \text{für } \zeta \ll f$$

geostrophe Näherung: stream function

$$u = -\frac{\partial\eta}{\partial y}\;,\quad v = \frac{\partial\eta}{\partial x}\;,\quad \text{also } \eta \equiv \psi \;✓$$

$$\text{also } \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} = \Delta\eta \quad \text{wie zuvor}$$

$$\text{shallow water: } \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Also $d\Pi/dt = 0$!

$$\boxed{\left(\frac{\partial}{\partial t} - \frac{\partial\eta}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial y}\right)\left(\overset{1}{\Delta\eta} - \overset{2}{\frac{f\eta}{D}} + \overset{3}{f} + \overset{4}{\frac{f h_B}{D}}\right) = 0}$$

Pedlosky Glg (3.12.25), S. 91

Linearisieren = weglassen aller η^2:

$$\boxed{\Delta\dot{\eta} - \frac{f}{D}\dot{\eta} + J(\eta, f + f h_B/D) = 0}$$

mit Jacobideterminante $J = \frac{\partial\eta}{\partial x}\frac{\partial}{\partial y}(f + f h_B/D)$

$$- \frac{\partial\eta}{\partial y}\frac{\partial}{\partial x}(f + f h_B/D)$$

damit 2-D Rossbywellen: sei $f = \text{const}$,

$$\Delta\dot{\eta} - \underset{f/D}{\underset{\parallel}{F}}\dot{\eta} + \frac{\partial\eta}{\partial x}F\frac{\partial h_B}{\partial y} - \frac{\partial\eta}{\partial y}F\frac{\partial h_B}{\partial x} = 0$$

sei $\eta = e^{i(kx + ly - \sigma t)}$. dann

$$-(k^2 + l^2 + F)(-i\sigma) = -ikF\frac{\partial h_B}{\partial y} + ilF\frac{\partial h_B}{\partial x}$$

$$\boxed{\sigma = -F\,\frac{k\,\partial h_B/\partial y - l\,\partial h_B/\partial x}{k^2 + l^2 + F}}$$

Dispersionsrelation (3.15.4) auf S. 99 in Pedlosky

Schließlich der eigentliche Rossbyfall: β-Effect:

$$f = f_0 + \beta y, \quad h_B = 0$$

$$\boxed{\Delta\dot{\eta} - F\dot{\eta} + \beta\frac{\partial\eta}{\partial x} = 0}$$

Glg (3.25.1) auf S. 144 in Pedlosky

($F = f/D$ = Coriolisfrequ. / Wassertiefe)

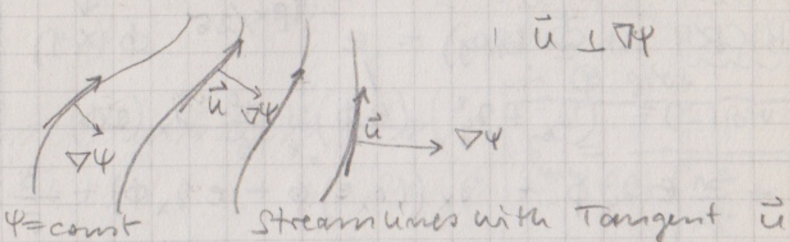
Neue Physik: Rossbywellen in Basins

Pedlosky S. 147

Vorbetrachtung: $u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$

d.h. $\vec{u}\cdot\nabla\psi = (-\psi_y, \psi_x)(\psi_x, \psi_y) = 0$

$\Rightarrow \vec{u} \perp \nabla\psi$



$$\boxed{\text{Streamline} \equiv \text{curve with const. } \psi}$$

No cross-boundary velocity component

$\equiv$ $\vec{u}$ is tangent to boundary

But $\vec{u}$ is tangent to ψ = const

$\Rightarrow$ $\boxed{\psi = \text{const along boundary}}$ but may be function of time!

Here η = const along boundary! (strange; geostrophic)

Assume $F = f/D \equiv 0$: small basin

$F \neq 0$ is very complicated! Flierl 1977

$$\boxed{\begin{aligned}\Delta\dot{\eta} + \beta\frac{\partial\eta}{\partial x} &= 0\\ \eta = 0 \quad &\text{on boundary}\end{aligned}}$$

Ansatz: $\eta(x,y,t) = \Phi(x,y)\, e^{-i\sigma t}$ gibt

$$\Delta\Phi + \frac{i\beta}{\sigma}\Phi_x = 0$$

is linear in ϕ, too, thus try

still there!

TRICK $\quad \Phi(x,y) = e^{-i\beta x/2\sigma}\,\phi(x,y)$

gives $0 = (\partial_x^2 + \partial_y^2)(e\phi) + \frac{i\beta}{\sigma}\partial_x(e\phi)$

$$= e\,\partial_y^2\phi + \partial_x\big((\partial_x e)\phi + e\,\partial_x\phi\big) + \frac{i\beta}{\sigma}(\partial_x e)\phi + \frac{i\beta}{\sigma}e\,\partial_x\phi$$

$$= e\,\partial_y^2\phi + (\partial_x^2 e)\phi + 2(\partial_x e)\partial_x\phi + e\,\partial_x^2\phi + \frac{i\beta}{\sigma}e\,\partial_x\phi + \frac{i\beta}{\sigma}(\partial_x e)\phi$$

$$= e\,\partial_y^2\phi - \frac{\beta^2}{4\sigma^2}e\phi - \cancel{2\frac{i\beta}{2\sigma}e\,\partial_x\phi} + e\,\partial_x^2\phi + \cancel{\frac{i\beta}{\sigma}e\,\partial_x\phi} - \frac{i\beta}{\sigma}\frac{i\beta}{2\sigma}e\phi$$

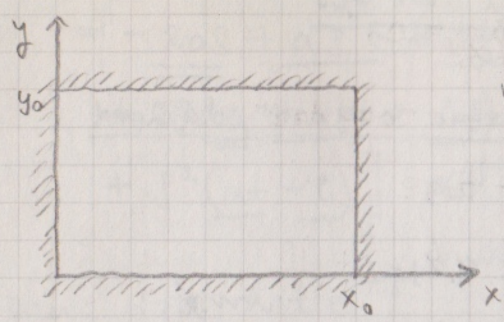
$$= \cancel{e}\,\partial_y^2\phi + \cancel{e}\,\partial_x^2\phi + \underbrace{\frac{\beta^2}{4\sigma^2}}_{-\frac14+\frac12=\frac14}\cancel{e}\phi$$

d.h. $\boxed{\Delta\phi + \lambda^2\phi = 0, \quad \lambda = \frac{\beta}{2\sigma}}$ ⊛

„Membrangleichung"

+ R.B. $\psi = \eta = \Phi = \boxed{\phi = 0 \text{ on boundary}}$

Rectangular basin:



$\phi_{mn}(x,y) = \sin\frac{m\pi x}{x_0}\sin\frac{n\pi y}{y_0}$

$m = 1,2,3,\ldots,\; n = 1,2,3,\ldots$

erfüllt die RB, einsetzen in ⊛ gibt

$$\boxed{\sigma_{mn} = \frac{(-)\beta}{2\pi\sqrt{\frac{m^2}{x_0^2} + \frac{n^2}{y_0^2}}}}$$

d.h. je feiner die Schwingung (d.h. je größer m, n) desto niedriger Frequenz! counter-intuitive

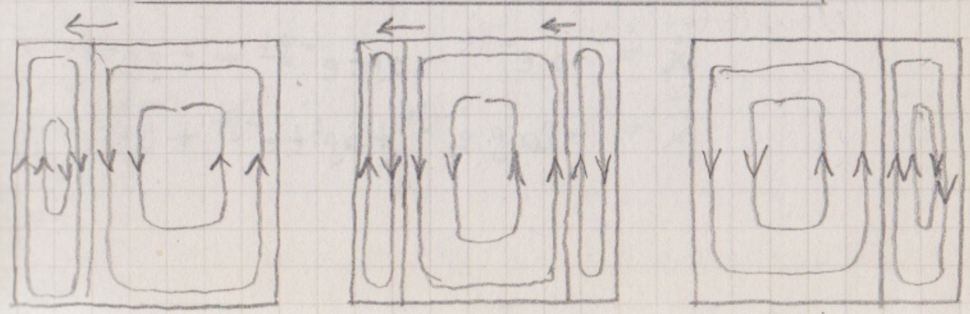
$$\eta(x,y,t) = \cos\left(\frac{\beta x}{2\sigma_{mn}} + \sigma_{mn}t\right)\sin\frac{m\pi x}{x_0}\sin\frac{n\pi y}{y_0}$$

carrier wave moves from right to left = westward

ist Glg. (3.25.16) S. 147 Pedlosky

mit Wellen(phasen)geschwindigkeit $\left(kx + \sigma t = 0,\; c = -\sigma/k\right)$

$$\boxed{c = -\frac{2\sigma_{mn}^2}{\beta} = -\frac{\beta}{2\pi^2\left(\frac{m^2}{x_0^2} + \frac{n^2}{y_0^2}\right)}}$$ (3.25.17) Pedlosky



$\sigma_{11}t = 0$ $\qquad$ $\sigma_{11}t = \pi/4$ $\qquad$ $\sigma_{11}t = \pi/2$

RESONANT TRIADS FOR ROSSBY WAVES

from Pedlosky p. 153 – 164

1) Preliminaries: harmonic oscillator, damped

$$F_{Hooke} = -kx$$

$$F_{Stokes} = -\eta \dot{x}$$

$$m\ddot{x} + \eta \dot{x} + kx = 0$$

$$\ddot{x} + \frac{\eta}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\rho \dot{x} + \omega_0^2 x = 0$$

Ansatz:

$$x = e^{\lambda t}$$

$$\dot{x} = \lambda e^{\lambda t}$$

$$\ddot{x} = \lambda^2 e^{\lambda t}$$ ohne i

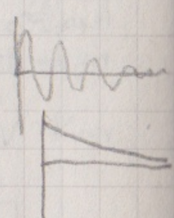
$$\rightarrow \quad \lambda^2 + 2\rho\lambda + \omega_0^2 = 0$$

$$\lambda_\pm = -\rho \pm \sqrt{\rho^2 - \omega_0^2}$$

$$= -\rho \mp i\sqrt{\omega_0^2 - \rho^2}$$

ρ small: $\quad x_\pm = e^{\mp i\sqrt{\omega_0^2 - \rho^2}\, t}\, e^{-\rho t}$

ρ large: $\quad x_\pm = e^{-\left(\rho \pm \sqrt{\rho_0^2 - \omega_0^2}\right)t}$



Aperiodischer Grenzfall: $\rho = \omega_0$, Doppellösung

Mathematik: $x = a t e^{-\rho t} + b e^{-\rho t}$

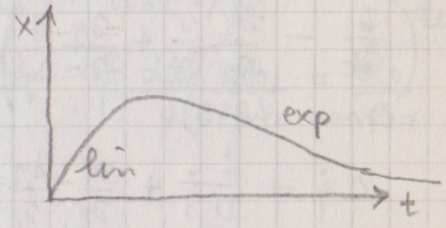
$$\dot{x} = a e^{-\rho t} - a\rho t e^{-\rho t} - b\rho e^{-\rho t}$$

$$\ddot{x} = -2a\rho e^{-\rho t} + a\rho^2 t e^{-\rho t} + b\rho^2 e^{-\rho t}$$

$$\ddot{x} + 2\rho\dot{x} + \rho^2 x = \left(\frac{d}{dt} + \rho\right)\left(\frac{d}{dt} + \rho\right) x$$

$$= \Big[-2a\rho + a\rho^2 t + b\rho^2 + 2\rho\left(a - a\rho t - b\rho\right) + \rho^2\left(at + b\right) \Big]\, e^{-\rho t} = 0 \quad \text{indeed}$$

new: $\quad x = t e^{-\rho t}$



Similarly for resonance amplitude in driven oscillator: amplitude $\sim t$ initially

Equation of motion for resonant triad 19.3.22

Pedlosky

Oberflächenwellen: 4-Resonanz

Rossbywellen: 3-Resonanz: $\vec{u}$ von Welle A advektiert $\nabla\times\vec{u}$ von Welle B

von vorher pot. vort. cons.:

$$\left(\frac{\partial}{\partial t} - \frac{\partial\eta}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial y}\right)\left(\Delta\eta - \frac{f}{D}\eta + f + f\frac{h_B}{D}\right) = 0$$

nimm mit:

$$\Delta\dot\eta - \underbrace{\frac{f}{D}}_{F}\dot\eta + \underbrace{\frac{\partial f}{\partial y}}_{\beta}\frac{\partial\eta}{\partial x} + \underbrace{\frac{\partial\eta}{\partial x}\frac{\partial\Delta\eta}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial\Delta\eta}{\partial x}}_{\text{Jacobideterminante}} = 0$$

$$\boxed{\Delta\dot\eta - F\dot\eta + \beta\eta_x + J(\eta,\Delta\eta) = 0} \qquad (3.26.1)$$

Pedlosky

$$\frac{\partial}{\beta\partial t}(\Delta\eta - F\eta) + \frac{\partial\eta}{\partial x} + \frac{1}{\beta}J(\eta,\Delta\eta) = 0$$

sei $\beta \gg 1$: Wellen auf Sub-Corioliskala

schnelle Zeit $\boxed{\tau = \beta t} \gg t$

$$\frac{\partial}{\partial\tau}(\Delta\eta - F\eta) + \frac{\partial\eta}{\partial x} + \frac{1}{\beta}J(\eta,\Delta\eta) = 0 \quad (*)$$

Gesucht ist Welle $a\,e^{i(kx+ly-\sigma t)} = a\overset{\text{phase}}{p}(t)$

Ansatz 1 — durch Wechselwirkung ändert sich Wellenamplitude, aber langsam!

„Skalen-theorie“

$$\eta = a\left(\frac{\tau}{\beta}\right)p(\tau)$$

Die DGL „kümmert“ sich bevorzugt um $\frac{\partial}{\partial\tau}p(\tau)$, Korrekturglieder $da(t)/dt$ ($t = \tau/\beta$) beschreiben langsame Entwicklung

Ansatz 2 — Störungstheoretisch, „Ordnungen“ in β

$$\boxed{\eta = a_0\left(\frac{\tau}{\beta}\right)p_0(\tau) + \frac{1}{\beta}\eta_1(\tau)}$$

Bem: Pedlosky fängt mit $\eta_0(\tau) + \frac{1}{\beta}\eta_1(\tau)$ an

und korrigiert später $a_0 \to a_0(\tau/\beta)$

langsame Zeit

einsetzen in (*) mit $\frac{da_0(\tau/\beta)}{d\tau} = \frac{1}{\beta}\frac{da_0(\tau/\beta)}{d(\tau/\beta)} = \frac{\mathring{a}_0}{\beta}$

$$a_0\Delta\dot p_0 - Fa_0\dot p_0 + a_0 p_{0x} + \frac{1}{\beta}\left(\Delta\dot\eta_1 - F\dot\eta_1 + \eta_{1x}\right) + \frac{1}{\beta}\mathring{a}_0(\Delta p_0 - Fp_0) + \frac{1}{\beta}J(\eta_0,\Delta\eta_0) = 0$$

d.h. nach Trennung von β^0 und β^{-1}:

$$\boxed{\begin{aligned} a_0\Delta\dot p_0 - Fa_0\dot p_0 + a_0p_{0x} &= 0 \\ \Delta\dot\eta_1 - F\dot\eta_1 + \eta_{1x} &= -\mathring{a}_0(\Delta p_0 - Fp_0) - J(\eta_0,\Delta\eta_0) \end{aligned}}$$

Jetzt Entwicklung von J! sei $\eta_0 = \sum_{j=1}^{3} a_j\cos\theta_j$

$J(\eta_0,\Delta\eta_0) = \eta_x\Delta\eta_y - \eta_y\Delta\eta_x$ (mit $\theta_j = k_jx + l_jy - \sigma_jt$)

$$= \sum_m\sum_n a_m a_n(k_n^2 + l_n^2)(k_nl_m - k_ml_n)\sin\theta_m\sin\theta_n$$

(n↔m) $$= -\sum_m\sum_n a_m a_n(k_m^2 + l_m^2)(k_nl_m - k_ml_n)\sin\theta_m\sin\theta_n$$

(add.) $$= -\frac{1}{2}\sum_{mn}\sum(K_m^2 - K_n^2)(k_nl_m - k_ml_n)\sin\theta_m\sin\theta_n$$

$(K_m^2 = k_m^2 + l_m^2$ usw.$)$

$$= -\sum_m \sum_n a_m a_n B_{mn} \cdot \left(\cos(\theta_m + \theta_n) - \cos(\theta_m - \theta_n) \right)$$

mit $\boxed{B_{mn} = \frac{1}{4} \left(K_m^2 - K_n^2 \right) \left(\vec{K}_m \times \vec{K}_n \right) \cdot \hat{z}}$

da $\vec{K}_m \times \vec{K}_n = (k_m \hat{x} + l_m \hat{y}) \times (k_n \hat{x} + l_n \hat{y})$

$= (k_m l_n - k_n l_m) \hat{x} \times \hat{y}$

$= (k_m l_n - k_n l_m) \hat{z}$ $\left(\begin{array}{l} -1 \text{ vom } \cos(\theta_m \pm \theta_n) \\ \text{wird hier hin gezogen} \end{array} \right)$

d.h.

$$\boxed{\begin{aligned} &\Delta \dot{\eta}_1 - F \dot{\eta}_1 + \eta_{1x} = \\ (*) \quad &= -\dot{a}_0 (\Delta p_0 - F p_0) \\ &+ \sum_m \sum_n a_m a_n B_{mn} \cos(\theta_m + \theta_n) \end{aligned}}$$

mit B_{mn} von oben, wobei manifest $B_{mn} = B_{nm}$

$\cos(\theta_m - \theta_n)$ gibt keine neue Info:

Im folgenden freie Wahl des gewünschten Vorzeichens.

So 20.3.22

Die rechte Seite in (*) stellt „forcing" term, für eine Schwingung auf der linken Seite mit

$$\boxed{\begin{aligned} k_r &= k_m \pm k_n \\ l_r &= l_m \pm l_n \\ \sigma_r &= \sigma_m \pm \sigma_n \end{aligned}}$$

Beachte $B_{mn} = 0$ für

$K_m^2 = K_n^2$: gleiche Wellenlänge

& $\vec{k}_m = \alpha \vec{k}_n$: parallele Wellenvektoren

Dies wird umgeschrieben zu

$$\boxed{\begin{aligned} k_m + k_n + k_r &= 0 \\ l_m + l_n + l_r &= 0 \\ \sigma(k_m, l_m) + \sigma(k_n, l_n) + \sigma(k_r, l_r) &= 0 \end{aligned}}$$

passende Vorzeichenwahl ist erlaubt

wobei letzte Zeile

$$\boxed{\frac{k_m}{k_m^2 + l_m^2 + F} + \frac{k_n}{k_n^2 + l_n^2 + F} + \frac{k_r}{k_r^2 + l_r^2 + F} = 0}$$

Heavy algebra: Longuet-Higgins & Gill (1967)

Oszillatortreibung mit $\theta_m + \theta_n$, also Ansatz

$$\eta_1 = \sum_{m,n} a_{1,mn} \sin(\theta_m + \theta_n)$$

Setze vorläufig $\dot{a}_0 = 0$, d.h. (*) gilt

$$\Big\{ -(\sigma_m + \sigma_n) \left(-\left[(k_m + k_n)^2 + (l_m + l_n)^2 + F \right] \right) + (k_m + k_n) \Big\} a_1 = a_m a_n B_{mn}$$

$-\sigma t$

$$a_{1mn} = \frac{a_m a_n B_{mn}}{\left[(k_m + k_n)^2 + (l_m + l_n)^2 + F \right] \Big[(\sigma_m + \sigma_n) - \underbrace{\frac{-(k_m + k_n)}{(k_m + k_n)^2 + (l_m + l_n)^2 + F}}_{= \sigma(k_m + k_n, l_m + l_n)} \Big]}$$

Resonant wenn wie oben

$$\boxed{\sigma(k_m, l_m) + \sigma(k_n, l_n) = \sigma(k_m + k_n, l_m + l_n)}$$

(Vorzeichenfreiheit)

$$= -\frac{k_m}{k_m^2 + l_m^2 + F}$$

Resonanztripel

ersetze m, n, r durch $1,2,3$ als eine resonant triad von vielen

Bisherige Ansätze

$$\eta_0 = \sum_{j=1}^{\infty} a_{0j} \cos\theta_j = \sum a_{0j} \rho_{0j}$$

$$\eta_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{1mn} \sin(\theta_m + \theta_n)$$

Die Bewegungsgleichungen sind

(0) $\cancel{a_{0j}}\Delta \dot{\rho}_{0j} - F \cancel{a_{0j}} \dot{\rho}_{0j} + \cancel{a_{0j}} \frac{\partial \rho_{0j}}{\partial x} = 0$ ist erfüllt

(1) $$\Delta\dot{\eta}_1 - F\dot{\eta}_1 + \eta_{1x} = \sum_j \mathring{a}_{0j}(\Delta\rho_{0j} - F\rho_{0j}) + \sum_m\sum_n a_{0m} a_{0n} B_{mn} \cos(\theta_m + \theta_n)$$

Gleichung (1) OHNE den Term $\mathring{a}_0$ ist erfüllt mit a_{1mn} von der letzten Seite (Resonanzbruch)

Jetzt neu: betrachte in (1) 3 Gleichungen für 1 Tripel m, n, r

$$\boxed{\eta_0 = a_{0m}\cos\theta_m + a_{0n}\cos\theta_n + a_{0r}\cos\theta_r}$$

in (1):

$$\Delta\dot{\eta}_1 - F\dot{\eta}_1 + \eta_{1x} =$$

$$= \mathring{a}_{0r}(k_r^2 + l_r^2 + F)\cos\theta_r + a_{0m}a_{0n}B_{mn}\cos(\theta_m + \theta_n)$$
$$+ \mathring{a}_{0m}(k_m^2 + l_m^2 + F)\cos\theta_m + a_{0n}a_{0r}B_{nr}\cos(\theta_n + \theta_r)$$
$$+ \mathring{a}_{0n}(k_n^2 + l_n^2 + F)\cos\theta_n + a_{0r}a_{0m}B_{rm}\cos(\theta_r + \theta_m)$$

(nebenstehende Seite)

Ist nur check, dass Skalenansatz okay, kein „remarkable result" wie bei Pedlosky ←

Jetzt: die rechte Seite ist die Treibung, verursacht Resonanz und Amplitudenwachstum. Forderung: soll verschwinden, damit bisheriger Störungsansatz korrekt, also: $\boxed{\text{rechte Seite} = 0}$! (willkür)

Lasse „0" weg und ersetze $m,n,r \to 1,2,3$

$$\boxed{\begin{aligned} \mathring{a}_1 + \frac{B_{23}}{K_1^2+F} a_2 a_3 &= 0 \\ \mathring{a}_2 + \frac{B_{31}}{K_2^2+F} a_3 a_1 &= 0 \\ \mathring{a}_3 + \frac{B_{12}}{K_3^2+F} a_1 a_2 &= 0 \end{aligned}}$$

Glg (3.26.33) S. 160 in Pedlosky

Übung: Energiedichte Rossbywelle ist $\frac{a^2}{4}(K^2+F)$

also $$\frac{d}{dt}(E_1 + E_2 + E_3) =$$
$$= \tfrac{1}{2} a_1 \mathring{a}_1 (K_1^2 + F) + \tfrac{1}{2} a_2\mathring{a}_2(K_2^2+F) + \tfrac{1}{2} a_3\mathring{a}_3(K_3^2+F)$$
$$= -\tfrac{1}{2} a_1 a_2 a_3 (B_{12} + B_{23} + B_{31}) = 0$$

Denn $4(B_{12} + B_{23} + B_{31}) =$

$$(K_1^2 - K_2^2)(\vec{K}_1\times\vec{K}_2) + (K_2^2 - K_3^2)(\vec{K}_2\times\vec{K}_3) + (K_3^2 - K_1^2)(\vec{K}_3\times\vec{K}_1)$$

$$= (K_1^2 - K_2^2)(\vec{K}_1\times\vec{K}_2) + (K_2^2 - K_3^2)(\vec{K}_1\times\vec{K}_2) + (K_3^2 - K_1^2)(\vec{K}_1\times\vec{K}_2) = 0$$

$\begin{pmatrix}\vec{K}_1 + \vec{K}_2 + \vec{K}_3 \\ = 0\end{pmatrix}$

also tauscht die Triade unter sich Energie aus.

weiterer Erhaltungssatz (der Rotation). Ihe (neu)

$$\frac{d}{dt}\left(K_1^2 E_1 + K_2^2 E_2 + K_3^2 E_3\right)$$
$$= \tfrac{1}{2} K_1^2 a_1 \dot a_1 (K_1^2+F) + \tfrac{1}{2} K_2^2 a_2 \dot a_2 (K_2^2+F) + \tfrac{1}{2} K_3^2 a_3 \dot a_3 (K_3^2+F)$$
$$= -\tfrac{1}{2} K_1^2 a_1 B_{23} a_2 a_3 - \tfrac{1}{2} K_2^2 a_2 B_{31} a_3 a_1 - \tfrac{1}{2} K_3^2 a_3 B_{12} a_1 a_2$$
$$= -\tfrac{1}{8} a_1 a_2 a_3 \Big(K_1^2 (K_2^2 - K_3^2)(\vec K_2 \times \vec K_3)\cdot \hat z + K_2^2 (K_3^2 - K_1^2)(\vec K_3 \times \vec K_1)\cdot \hat z + K_3^2 (K_1^2 - K_2^2)((\vec K_1 \times \vec K_2)\cdot \hat z)\Big)$$
$$\underset{\substack{(\vec K_1 + \vec K_2 + \vec K_3 \\ = 0)}}{=} -\tfrac{1}{8} a_1 a_2 a_3 \left(K_1^2(K_2^2-K_3^2) + K_2^2(K_3^2-K_1^2) + K_3^2(K_1^2-K_2^2)\right)(\vec K_1\times\vec K_2)\cdot\hat z$$
$$= 0.$$

Also $0 = K_1^2 \dot E_1 + K_2^2 \dot E_2 + K_3^2 \dot E_3$
$$= K_1^2 \dot E_1 + K_2^2 \dot E_2 - K_3^2(\dot E_1 + \dot E_2)$$
$$= (K_1^2 - K_3^2)\dot E_1 + (K_2^2 - K_3^2)\dot E_2$$

Annahme (OEdA): Energie von $\underbrace{1 \text{ und } 2}_{\text{W.Wirkung}}$ nach 3

also $\operatorname{sgn} \dot E_1 = \operatorname{sgn} \dot E_2$

also $\operatorname{sgn}(K_1^2 - K_3^2) = -\operatorname{sgn}(K_2^2 - K_3^2)$

$\operatorname{sgn}(K_1 - K_3) = -\operatorname{sgn}(K_2 - K_3)$

also entweder $K_1 > K_3,\ K_2 < K_3$ oder $K_1 < K_3,\ K_2 > K_3$

$\boxed{K_1 > K_3 > K_2} \vee \boxed{K_1 < K_3 < K_2}$

Die Welle, in die Energie transferiert wird, muss eine kurzwelligere und eine langwelligere Ursprungswelle haben

Entsprechend: von 1 nach $\underbrace{2 \text{ und } 3}_{\text{Zerfall}}$

also $\operatorname{sgn} \dot E_2 = \operatorname{sgn} \dot E_3$

$$0 = K_1^2 \dot E_1 + K_2^2 \dot E_2 + K_3^2 \dot E_3$$
$$= (K_2^2 - K_1^2)\dot E_2 + (K_3^2 - K_1^2)\dot E_3$$
$$\rightarrow \operatorname{sgn}(K_2 - K_1) = -\operatorname{sgn}(K_3 - K_1)$$

also $K_2 > K_1,\ K_3 < K_1$ oder $K_2 < K_1,\ K_3 > K_1$

$\boxed{K_3 < K_1 < K_2} \vee \boxed{K_3 > K_1 > K_2}$

Welle kann Energie abgeben nur an 2 andere Wellen, eine kurzwelliger, eine langwelliger

Noch etwas Kinematik des Austauschs:

Sei $a_1 \gg a_2 \longrightarrow a_3$

$$\dot a_1 + \frac{B_{23}}{K_1^2 + F} \underbrace{a_2 a_3}_{\text{klein von Ordnung 2}} = 0 \quad \text{d.h. } \dot a_1 \approx 0$$

$$\left.\begin{aligned} \dot a_2 + \frac{B_{31}}{K_2^2+F} a_1 a_3 &= 0 \\ \dot a_3 + \frac{B_{12}}{K_3^2+F} a_1 a_2 &= 0 \end{aligned}\right\}$$

$$\rightarrow \ddot a_2 \approx -\frac{B_{31}}{K_2^2+F} a_1 \dot a_3 = \frac{B_{12} B_{13}}{(K_2^2+F)(K_3^2+F)} a_1^2 a_2$$

also exponentielles Wachstum von a_2 und a_3

„The pulsation is perpetual, each member of the triad first receiving and then returning energy to the others." Pedlosky S. 163

Trivial: $\dot{E}_1 + \dot{E}_2 + \dot{E}_3 = 0$ nur möglich für folgende Vorzeichenkombinationen

$\oplus - -$ und $+ \ominus +$ & ihre Permutationen

In beiden Fällen (& ihren Perm.s) gibt es immer eine Welle die empfängt oder spendet.

Weiter nach Longuet-Higgins & Gill 1967

WIND-DRIVEN OCEANIC CIRCULATON 21.3.22

= Pedlosky Chap. 5

Starke Meeresströmung von Ost nach West um Äquator, von $-10°$ bis $10°$ geograph. Breite

Ursache sind die trade winds!

In mittleren Breiten sind Strömungen dagegen von West nach Ost, aufgrund der westerly winds (d.h. Wind von Westen)

Einfachste Bewegungsgleichung

$$\boxed{\vec{v} = \nabla \times \vec{\tau}} \quad \text{Sverdrup relation}$$

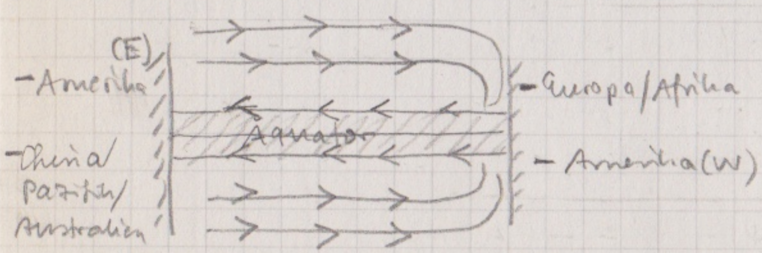
mit $\boxed{\vec{\tau} = C \rho_a u_a^2 \hat{u}_a}$ Empirical law for wind stress $\vec{\tau}$

ρ_a = air density, u_a = wind speed, $\hat{u}_a$ = ..direction
C = empirical coefficient (not constant!)

Annahmen in der Herleitung:

$\beta \gg 1$, $Re \cdot \beta \gg 1$

Mit dieser Gleichung allein (!) erhält man:



Welander 1959 with many details

Mehr Details: Munk Layer

Free inertial modes = Pedlosky Sect. 5.10 p305

Vorticity equation with everything neglected except variation of Coriolis parameter $f = f_0 + \beta y$:

$$\boxed{\vec{u} \cdot \nabla \left(c^{2} \zeta + y \right) = 0} \quad \circledast$$

ζ = vertical component of $\vec{\omega}$, c = constant

Geostrophic flow, $\zeta = \Delta \Psi$ wegen $u = -\partial_y \Psi$, $v = \partial_x \Psi$

Erinnerung: $\vec{u} \cdot \nabla \Psi = u \partial_x \Psi + v \partial_y \Psi =$

$$= -\partial_y \Psi \partial_x \Psi + \partial_x \Psi \partial_y \Psi = 0$$

d.h., da Ψ = const entlang Ψ-Höhenlinien:

$\vec{u} \cdot \nabla$ ist Ableitung entlang Ψ-Höhenlinien
d.h. $\vec{u}$ ist parallel zu —"—

Also kann man $\circledast$ schreiben also

$$\boxed{c^2 \Delta \Psi + y = f(\Psi)}$$

da auch die Ableitung von $f(\Psi)$ entlang Ψ-Höhenlinien verschwindet, $f(\Psi)$ = const für Ψ = const

WÄHLE $\boxed{f(\Psi) = \frac{\Psi}{A^2}}$ mit Konstante A

Also Randwertproblem

$$\boxed{A^2 c^2 \Delta \Psi - \Psi = -y A^2}$$

mit Ψ = const für $x = 0$, $x = l$, $y = 0$, $y = 1$.

Annahme: im Innern des Bassins ist $\Delta\Psi \approx 0$,

also $\Psi_I = A^2 y$ = uniform westward flow.

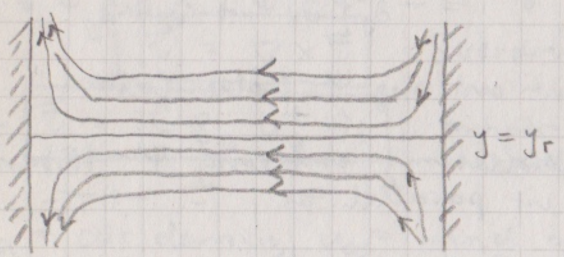
An den Rändern nimmt man $\Delta\Psi$ mit, um RB zu erfüllen

Diese Idee aus boundary layer theory

Vertikale Ränder $x=0$ und $x=l$:

$$\frac{\Psi}{A^2} = y - (y-y_r)\left[e^{-x/Ac} + e^{-(l-x)/Ac}\right]$$

mit freier Konstante y_r $\left[\text{for } e^{-x/Ac} \ll 1\right]$



gibt $u = -\Psi_y$

also $\frac{u}{A^2} = -1$ Innen

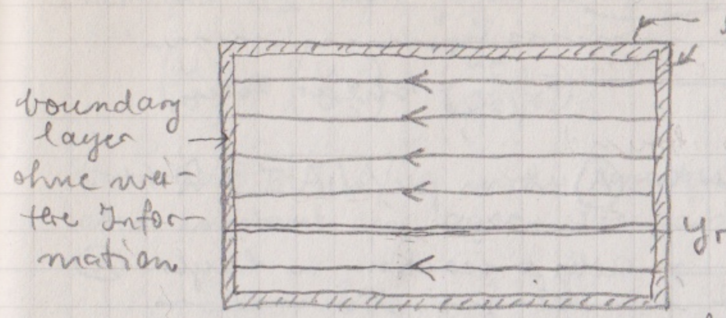
und $\frac{u}{A^2} = -1+1=0$ für $x=0$ u. $x=l$

(Brachial-)Idee: schließe nun oben durch „eine einzige" horizontale Rückstromlinie ab, d.h. $\Psi = \Psi(y)$, damit nur u, kein v. Passende Wahl:

$$\boxed{\frac{\Psi}{A^2} = y - (y-y_r)\left[e^{-x/Ac} + e^{-(l-x)/Ac}\right] + y_r\, e^{-y/Ac} - (1-y_r)\, e^{-(1-y)/Ac}}$$

Achtung, dies macht kein $u=+1$, sondern erzeugt nur boundary layer bei $y=0$ und 1 mit Dicke Ac, in dem Rückströmung möglich.

Vorfaktoren y_r und $1-y_r$ regulieren den Massenfluss proportional zur Breite der Streifen, eben y_r und $1-y_r$.



Fofonoff 1954

Pedlosky p 307: „free mode" → „fundamentally nonlinear mode"

Wind-driven flows

Gill S. 317 ff

29./30.03.2022

Annahmen: 1) uniform rotation of stratified fluid about a vertical axis = f-plane

2) horizontal pressure gradient = wind

Wind bewirkt direkten response in dünner Oberflächenschicht des Meeres = Ekman transport

Konkret: upper mixed layer, 10 - 100 m Tiefe

Zum Vergleich: Atmosphärenbewegung fast nur durch Sonnenheizung: Auftriebskräfte

Fakt: main oceanic currents are wind driven

Mathematik der Scherkräfte (Reibung, Wind)

Gill S. 320

wind AIR

1) ← (F_x, F_y) shear force

thin! surface layer width δz

layer (A) WATER

3) → actio = reaction in layer (A)

2) ← $\left(F_x - \frac{\partial F_x}{\partial z}\delta z,\ F_y - \frac{\partial F_y}{\partial z}\delta z\right)$

" layer (B) induced shear force on layer below

total force on layer (A):

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} + \begin{pmatrix} -F_x + \partial F_x/\partial z\ \delta z \\ -F_y + \partial F_y/\partial z\ \delta z \end{pmatrix} = \begin{pmatrix} \partial F_x/\partial z \\ \partial F_y/\partial z \end{pmatrix}\delta z$$

Diese Kraft wird sich offenbar linear mit der Oberfläche skalieren, auf dz sie nicht. Schreibe also

$$F_x = X\ \delta x\,\delta y \qquad F_y = Y\ \delta x\,\delta y$$

mit "Scherdruck" (X, Y)

Also
$$\delta m \cdot du/dt = \partial X/\partial z\ \delta x\,\delta y\,\delta z$$
$$\delta m \cdot dv/dt = \partial Y/\partial z\ \delta x\,\delta y\,\delta z$$

Also
$$-fv \quad \boxed{du/dt = \rho^{-1}\ \partial X/\partial z} \quad -\rho^{-1}\ \partial p/\partial x$$
$$+fu \quad \boxed{dv/dt = \rho^{-1}\ \partial Y/\partial z} \quad -\rho^{-1}\ \partial p/\partial y$$

Now, not for X and Y !

Der vertikale Bereich mit signifikantem $\partial X/\partial z$, $\partial Y/\partial z$ heißt Ekman-layer (1905)

Betrachte nur den Geschwindigkeitsbeitrag im layer:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \begin{pmatrix} u_e \\ v_e \end{pmatrix}$$

Natürlich ist (u_0, v_0) nicht unabhängig von z, also konstant im layer. Linearisiert:

$$\frac{\partial u_e}{\partial t} - f v_e = \frac{1}{\rho}\frac{\partial X}{\partial z}$$
$$\frac{\partial v_e}{\partial t} + f u_e = \frac{1}{\rho}\frac{\partial Y}{\partial z}$$

EKMAN LAYER

Integriere dies $\int dz$, über den gesamten Ekman layer: an dessen Boden $(X, Y) = (0, 0)$ und an Oberfläche $(X, Y) = (X_s, Y_s)$

sei $U_e = \int u_e\,dz$, $V_e = \int v_e\,dz$

Also
$$\boxed{\begin{aligned} \frac{\partial U_e}{\partial t} - f V_e &= \frac{1}{\rho} X_s \quad (1) \\ \frac{\partial V_e}{\partial t} + f U_e &= \frac{1}{\rho} Y_s \quad (2) \end{aligned}}$$

(1) + i(2): $\frac{\partial}{\partial t}\underbrace{(U_e + iV_e)} + if\underbrace{(U_e + iV_e)} = \frac{X_s}{\rho}$

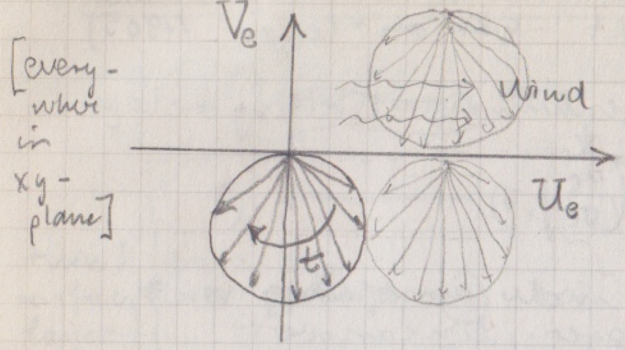
für $Y_s = 0$ — dieselbe Zahl $\in \mathbb{C}$! →

Gold (1908)
$$\boxed{U_e + iV_e = \frac{X_s}{i\rho f}\left(1 - e^{-ift}\right)}$$

Davon Real- und Imaginärteil nehmen gibt

$$\frac{\rho f}{X_s} \cdot U_e = \sin ft$$
$$\frac{\rho f}{X_s} \cdot V_e = \cos ft - 1$$

Gill S. 322

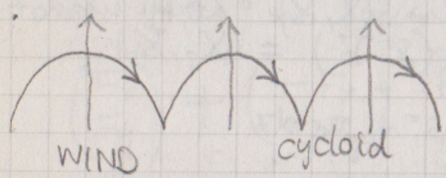


At first, ocean starts to flow with wind in x-direction, then flows to the right (north hemi) due to Coriolisforce

Antizyclonic rotation

$$\begin{pmatrix} U_e \\ V_e \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\text{constant transport normal to the wind}} + \underbrace{\begin{pmatrix} \sin ft \\ \cos ft \end{pmatrix}}_{\text{anticyclonic rotation (around inertial circles)}}$$

⊛: linear + circular = cycloidal (c.f. bicycle pedal)



⊛ to the definition of <u>this</u> cycloid

Dazu (natürlich) viele Beobachtungsdaten

<u>Innerer, vertikaler Aufbau</u>

<u>des Ekman layer</u>   Pedlosky p. 186

Ekman layer is about FRICTION; SHEAR at surface

Ekman layer = friction layer (p. 185)

Das folgende nur bzgl (z), keine Zeit (t)

vertikale Variation im Ekman layer

$$\left.\begin{aligned} -fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \\ fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} \end{aligned}\right\}\ \text{geostrophic + friction} \quad ⊛$$

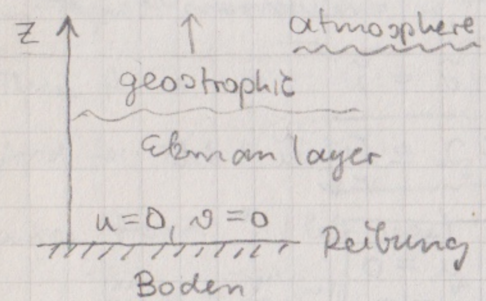
$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad \Big\}\ \text{hydrostatic}$$

beachte mal wieder shallow water:

$$\frac{\partial}{\partial z}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\frac{\partial p}{\partial z} = -\rho\frac{\partial g}{\partial x} = 0$$

$$\frac{\partial}{\partial z}\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\frac{\partial p}{\partial z} = -\rho\frac{\partial g}{\partial y} = 0$$

Der geostrophe Term $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$ ist höhenunabhängig, und so also das von ihm verursachte $(u,v) = (u,v)(x,y)$



(1) Für $z \to +\infty$

$u = U$
$v = 0$
$w = 0$

by choice of coordinate system $\hat{x}, \hat{y}$, where U is geostrophic velocity LOCAL!

For $z \to +\infty$, ⊛ becomes

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$fU = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

But $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$ are independent of z, thus these eqs. hold for all z!

Writing thus $\quad u = U + \tilde{u}$, $\quad v = \tilde{v}$ $\quad$ ⊛ becomes

$$-f\tilde{v} + \boxed{0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\frac{\partial^2 U}{\partial z^2} + \nu\frac{\partial^2 \tilde{u}}{\partial z^2}$$

$$f\tilde{u} + \boxed{fU = -\frac{1}{\rho}\frac{\partial p}{\partial y}} + \nu\frac{\partial^2 \tilde{v}}{\partial z^2}$$

cancels out
at all z

Thus

$$\boxed{\begin{aligned} f\tilde{u} &= \nu\frac{d^2\tilde{v}}{dz^2} \\ ¶ \quad -f\tilde{v} &= \nu\frac{d^2\tilde{u}}{dz^2}\end{aligned}}$$

where $\tilde{u}, \tilde{v}$ are functions <u>now</u> of z only, since this is the only differential appearing

$\tilde{u}, \tilde{v}$ are deviations from geostrophic flow in Ekman layer.

Eliminate $\tilde{v}$:

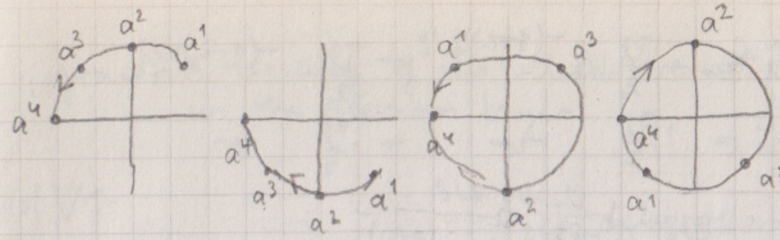
$$\frac{d^4\tilde{u}}{dz^2} = -\frac{f}{\nu}\frac{d^2\tilde{v}}{dz^2} = -\frac{f}{\nu}\frac{f}{\nu}\tilde{u} \quad \text{d.h.}$$

$$\boxed{\tilde{u}'''' + \frac{f^2}{\nu^2}\tilde{u} = 0}$$

try $\tilde{u} = \exp\left(a\frac{z}{\sqrt{\nu/f}}\right)$ mit $a \in \mathbb{C}$ und $\sqrt{\frac{\nu}{f}} = \tilde{\delta}$

$$\tilde{u}'''' + \frac{f^2}{\nu^2}\tilde{u} = \left(a^4\frac{f^2}{\nu^2} + \frac{f^2}{\nu^2}\right)\tilde{u} = 0$$

d.h. $a^4 = -1$ hat 4 komplexe Lösungen



$a = \frac{1+i}{\sqrt{2}}$ $\quad a = \frac{1-i}{\sqrt{2}}$ $\quad a = \frac{-1+i}{\sqrt{2}}$ $\quad a = \frac{-1-i}{\sqrt{2}}$

$45°\cdot 4 = 180°$ $\quad -45°\cdot 4 = -180°$ $\quad 135°\cdot 4 = 540 = 360+180$

Ziehe $\sqrt{2}$ in δ:

$$\delta = \sqrt{\frac{2\nu}{f}} = \sqrt{\frac{\nu}{f/2}}$$ ← cancels factor 2 in 1

$$\tilde{u} = c_1 e^{(1+i)z/\delta} + c_2 e^{(1-i)z/\delta} + c_3 e^{-(1+i)z/\delta} + c_4 e^{-(1-i)z/\delta}$$

renamed, compared to above

$C_1 = C_2 = 0$ since else $\tilde{u} \to \infty$ for $z \to \infty$.

Then for $z \to \infty$: $\tilde{u} = \tilde{v} = 0$ automatically

And for $z \to 0$: $\boxed{\tilde{u} = C_3 + C_4, \quad \tilde{v} = -ic_3 + ic_4}$

since from ¶ (left page):

$$\tilde{v} = -\frac{\nu}{f}\tilde{u}'' = -\frac{\nu}{f}(1+i)^2\frac{f}{2\nu}\cdot c_3 e^{-(1+i)z/\delta} - \frac{\nu}{f}(1-i)^2\frac{f}{2\nu}\cdot c_4 e^{-(1-i)z/\delta}$$

mit $\left.\begin{aligned}(1+i)^2 &= 2i \\ (1-i)^2 &= -2i\end{aligned}\right\}$ $\tilde{v} = -ic_3 e^{-(1+i)z/\delta} + ic_4 e^{-(1-i)z/\delta}$

Thus, since $0 = u(0) = U + \tilde{u} \to \tilde{u} = -U$

$0 = v(0) = \tilde{v} \to \tilde{v} = 0$

$\to \boxed{c_3 = c_4 = -U/2}$

Thus $\tilde u = -\frac{U}{2}\, e^{-(1+i)z/\delta} - \frac{U}{2}\, e^{-(1-i)z/\delta}$

$\tilde v = i\frac{U}{2}$ —"— $- i\frac{U}{2}$ —"—

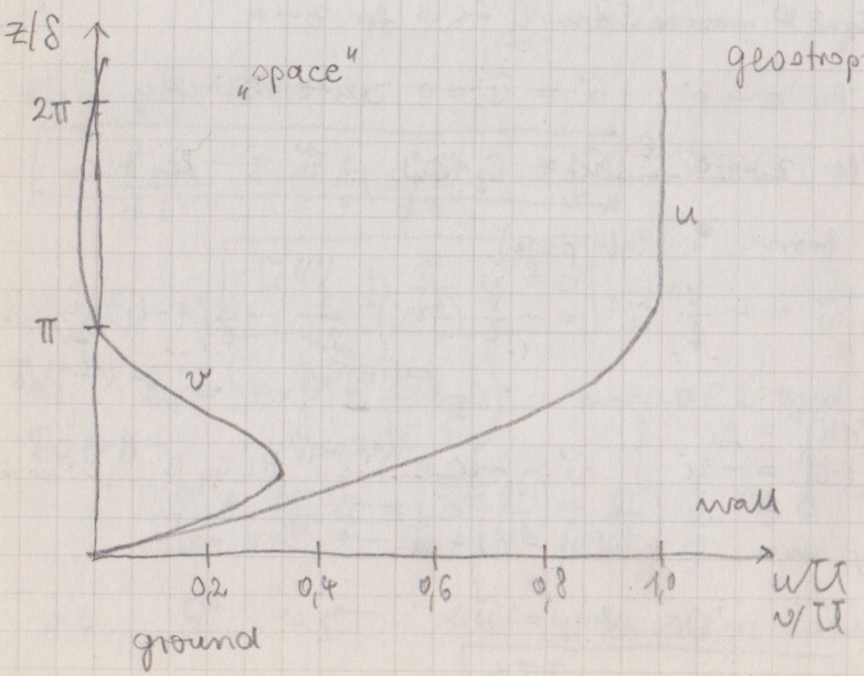
$$\tilde u = -\frac{U}{2}\, e^{-z/\delta}\left(\overbrace{e^{-iz/\delta} + e^{iz/\delta}}^{2\cos z/\delta}\right)$$

$$\tilde v = i\frac{U}{2}\, e^{-z/\delta}\left(\underbrace{e^{-iz/\delta} - e^{iz/\delta}}_{-2i\sin z/\delta}\right)$$

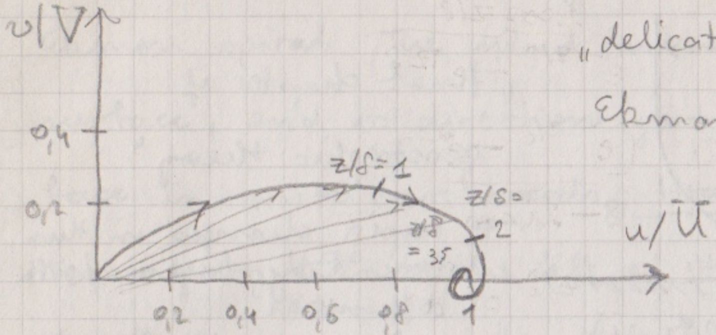
$$\tilde u = -U\, e^{-z/\delta} \cos\frac{z}{\delta}$$

$$\tilde v = U\, e^{-z/\delta} \sin\frac{z}{\delta}$$

$$\boxed{\begin{aligned} u &= U\left[1 - e^{-\frac{z}{\delta}}\cos\frac{z}{\delta}\right] \\ v &= U\, e^{-\frac{z}{\delta}}\sin\frac{z}{\delta} \end{aligned}}$$



„graceful turning of the velocity vector" (Pedlosky p 181)
in the Ekman layer

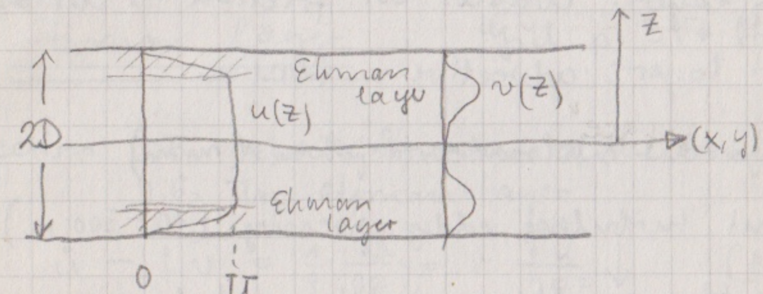


„delicate spiral"

Ekman spiral

Ekman layer is important for atmosphere and ocean for dissipation of kinetic energy.

Pedlosky diskutiert noch (p 185 ff) geostrophic flow with wall friction at walls $z = \pm 1$; top & bottom



(p. 187)

als Lösung von

$$\boxed{E^2 v'''' + 4v = 0}$$

$v = v(z)$, RB
$v(-1) = v(1) = 0$

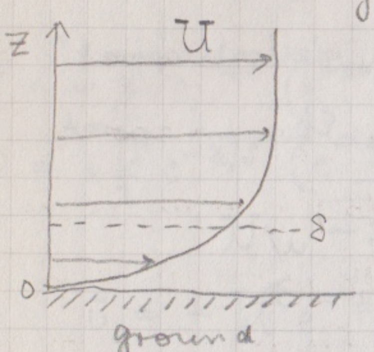
nämlich $v = A \sinh(kz) \sin(kz) + B \cosh(kx) \cos(kz)$

and Ekman number $\boxed{E = \frac{2\nu}{fD^2}} \ll 1$

and $k = \frac{D}{\delta} = \frac{1}{\sqrt{E}}$

Ekman layer   Vallis p. 201 – 211

E.l. is a boundary layer:



„final chapter of geostrophic theory"

→ rapid changes in thin boundary layers with differential

$\underbrace{\nu}_{\text{small}} \underbrace{\Delta v}_{\text{large}}$!

small scale ⟷ highest differential order

Ekman layer: Coriolis vs. friction → Ekman

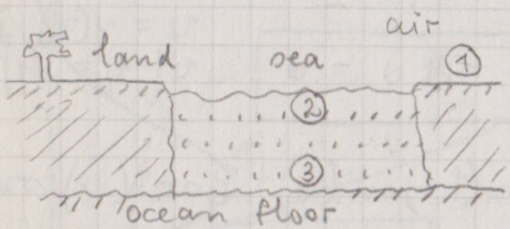
Prandtl layer: advective vs. friction

Achtung: not molecular friction (mm)

but turbulent eddy viscosity (10 – 300 m)

↓

no theory, thus simple models



three Ekman layers

assumption: friction is $\nu \frac{\partial^2 u}{\partial z^2}$

„Assume that the geostrophic current is eastwards. Then the solution is the now-famous Ekman spiral. The wind falls to zero at the surface, and its direction just above the surface is northeastwards; that is, it is rotated by 45° to the left of its direction in the free atmosphere." (Vallis, p. 206)

This Ekman spiral is observed in the ocean since the 1980 (difficult to measure (u,v))

Ekman layer   Gill p. 317 ff again

Section 9.5 velocity structure of the boundary layer
9.6. The Ekman layer   p. 328 ff

(1) $\dot{u} - f v = \frac{1}{\rho} \frac{\partial X}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2}$ (ρ = const)

(2) $\dot{v} + f u = \frac{1}{\rho} \frac{\partial Y}{\partial z} = \nu \frac{\partial^2 v}{\partial z^2}$

in Ekman (1905) layer. (1) + i(2):

$$\left(\frac{\partial}{\partial t} + if\right)(u + iv) = \nu \frac{\partial^2}{\partial z^2}(u + iv)$$

mit stationäre Lösung

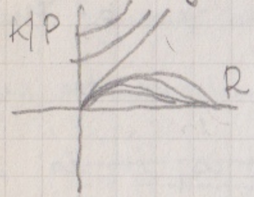
$$\boxed{u + iv = -(u_g + i v_g)\, e^{-(1+i) z/\delta}}$$

geostrophic

$\delta = \sqrt{\frac{2\nu}{f}}$

COMBINED ROSSBY – GRAVITY WAVES 1.4.22

we had, many pages back, combined Rossby + Kelvin/Poincaré waves



Now more directly: Rossby + (internal) gravity waves

(1) Vallis S. 298 ff : Rossby + shallow water waves

Rotating shallow-water equations

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} - f v &= - g' \frac{\partial \eta}{\partial x} && (1) \\ \frac{\partial v}{\partial t} + f u &= - g' \frac{\partial \eta}{\partial y} && (2) \\ \frac{\partial \eta}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 && (3) \end{aligned}}$$

η = free surface height

H = reference depth of fluid

g' = reduced gravity

z-Vorticity $\zeta = v_x - u_y \quad (= \Delta \psi)$

Assume $\boxed{f = \beta y}$ β-plane approximation

$(2)_x - (1)_y$:

$$\frac{\partial^2 v}{\partial t \, \partial x} - \frac{\partial^2 u}{\partial t \, \partial y} + \beta v + f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

using formula for ζ and (3):

$$\frac{\partial \zeta}{\partial t} - \frac{f}{H} \frac{\partial \eta}{\partial t} + \beta v = 0$$

linearized P.V. conservation

$$(4) \quad \boxed{\frac{\partial}{\partial t} \left(\zeta - \frac{f}{H} \eta \right) + \beta v = 0}$$

Vallis p. 304

Now the 1-equation trick: (ü = überlebt)

(I) $\frac{f}{g'H} \partial_t (1)$: $\quad \frac{f}{gH} u_{tt} - \frac{f^2}{g'H} v_t = - \frac{f}{H} \eta_{xt}$

(II) $\frac{1}{g'H} \partial_{tt} (2)$: $\quad \frac{1}{g'H} v_{ttt} + \frac{f}{g'H} u_{tt} = - \frac{1}{H} \eta_{ytt}$

(III) $\frac{1}{H} \partial_{ty} (3)$: $\quad \frac{1}{H} \eta_{tty} + u_{xyt} + v_{yyt} = 0$

(IV) $\partial_x (4)$: $\quad v_{xxt} - u_{xyt} - \frac{f}{H} \eta_{xt} + \beta v_x = 0$

Thus (I) − (II) − (III) − (IV) gives

$$\boxed{0 = - \frac{1}{g'H} \frac{\partial^3 v}{\partial t^3} + \frac{f^2}{g'H} \frac{\partial v}{\partial t} - \frac{\partial}{\partial t} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \beta \frac{\partial v}{\partial x}}$$

$g'H = c^2$ as usual. Vallis (8.2) p. 298

Note that $f = \beta y$ is not constant!

Strange but common assumption:

assume f = const afte differentiation of $f = \beta y$

Then the DGL has constant coefficient, & is linear

Solution thus $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\boxed{\omega \cdot \left(\frac{\omega^2 - f^2}{c^2} - k^2 - l^2 \right) - \beta k = 0} \quad \text{cubic!}$$

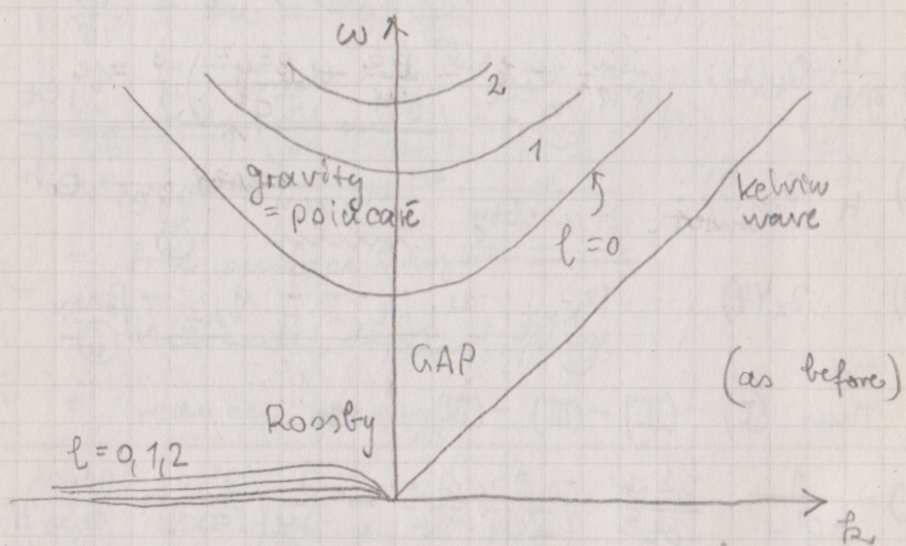
oder, indem man $f = 1 = c$ normiert,

$$\boxed{\omega^3 - \omega\left(1 + k^2 + l^2\right) - \beta k = 0} \quad \circledast$$

mit einem Parameter $\beta = 0.01 - 0.1$

ocean – atmosph.

($c = 200\,\frac{m}{s}$, $L = 1000$ km)



Trick: consider $\circledast$ as quadratic eq in k,

$$k = -\frac{\beta}{2\omega} \pm \left[\frac{\beta^2}{4\omega^2} + \omega^2 - l^2 - 1 \right]^{1/2}$$

Note that for $\omega \ll 1 = f$ we have Rossby waves,

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1 = k_d^2} \qquad \underline{\text{Rossby}}$$

whereas for $\omega \gg 1$, and $\beta = 0$ we have Poincaré waves,

$$\omega^2 = 1 + k^2 + l^2 \equiv f^2 + c^2\left(k^2 + l^2\right) \qquad \underline{\text{Poincaré}}$$

Nun ohne die Approx f = const

Koeffizient βy in der DGL

Diese immer noch linear bzgl. x, also

$$v(x, y, t) = f(y)\, e^{i(kx - \omega t)}$$

In DGL auf vorletzter Seite einsetzen gibt

Function f, not Coriolis accel. →

$$\boxed{f'' + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2} \right) f = 0}$$

↑ y-Abhängigkeit

unnormiert; ACHTUNG, mit $\boxed{\beta = 1}$

$$\boxed{f'' + \left(\omega^2 - k^2 - \frac{k}{\omega} - y^2 \right) f = 0}$$

Trick: setze $f(y) = g(y)\, \exp(-y^2/2)$. Damit wird

$$\boxed{g'' - 2yg' + \lambda g = 0} \qquad \text{HERMITE diff. eq.}$$

mit $\lambda = \omega^2 - k^2 - \frac{k}{\omega} - 1$

Hermite-Glg hat Lsg nur für $\lambda = 0, 2, 4, 6, \ldots = 2m$

Lösungen sind die Hermitepolynome

$$H_0 = 1$$
$$H_1 = 2y$$
$$H_2 = 4y^2 - 2$$
$$H_3 = 8y^3 - 12y$$
$$m \quad H_4 = 16y^4 - 48y^2 + 12 \quad \text{usw.}$$

also $$v_m(x,y,t) = H_m(y)\, e^{-y^2/2}\, e^{i(kx-\omega t)}$$

Dispersionsrelation aus

$$\lambda = \omega^2 - k^2 - \frac{k}{\omega} - 1 = 2m$$

d.h. $$\omega^2 - k^2 - \frac{k}{\omega} = 2m+1$$

vollständig $$\boxed{\omega^2 - c^2k^2 - \beta c^2 \frac{k}{\omega} = (2m+1)\beta c}$$

ist wieder kubische Gleichung.

Direktvergleich mit Rechnung für konstantes f:

$$\boxed{\begin{array}{ll} f=\text{const}: & \omega^3 - \omega(1+k^2+l^2) - \beta k = 0 \\ f=\beta y: & \omega^3 - \omega\left(1+k^2+(2m+1)\beta\right) - \beta k = 0 \end{array}}$$

$l \in \mathbb{R}$ beliebig, aber $l^2 \geq 0$ bisher

$(2m+1)\beta$ für $m=0$: $\beta > 0$ jetzt

$m = -1$ Kelvin waves

$m = 0$ mixed Rossby-gravity

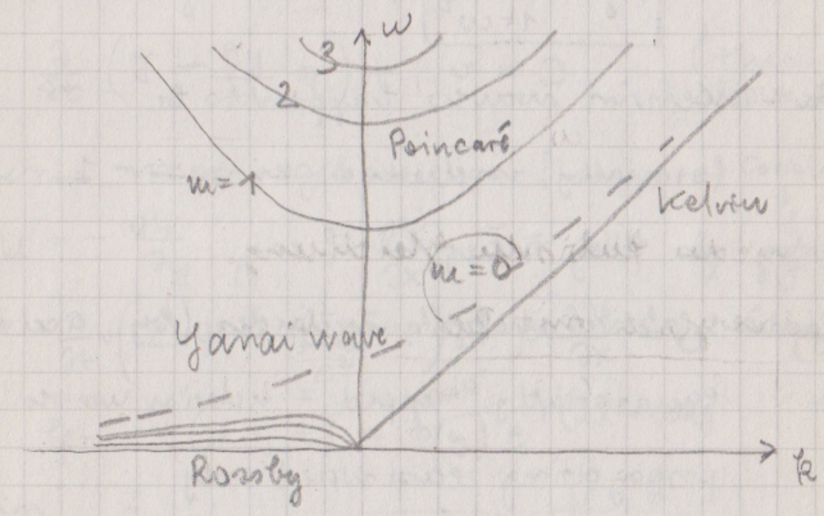
$m = 1,2,3,\ldots$ separated Rossby | gravity waves

gravity waves ($\omega \to \infty$) $$\omega_g^2 = k^2 + 2m + 1$$

Rossby waves ($\omega \to 0$) $$\omega_R = -\frac{k}{k^2 + 2m + 1}$$

Gap (again): $$\min(\omega_g) = \sqrt{2m+1}$$
$$\max(\omega_R) = -\frac{1}{2\sqrt{2m+1}}$$

d.h. $$\boxed{\frac{\min \omega_g}{\max \omega_R} = 2(2m+1)} \qquad m = 0,1,2,\ldots$$



Put $m=0$: Yanai wave (1966) = Rossby-gravity wave

$$\omega^2 - k^2 - \frac{k}{\omega} - 1 = 0, \quad k^2 + \frac{1}{\omega}k + 1 - \omega^2 = 0$$

$$k = -\frac{1}{2\omega} \pm \sqrt{\frac{1}{4\omega^2} - (1-\omega^2)} = -\frac{1}{2\omega} \pm \sqrt{\left(\omega - \frac{1}{2\omega}\right)^2}$$

d.h. $\boxed{k_1 = -\omega}$, $\boxed{k_2 = \omega - \frac{1}{\omega}}$

Vallis p. 310

k_1 is spurious. k_2 is a quadratic equation,

$$\omega_2 = \frac{k_2}{2} \pm \sqrt{\frac{k_2^2}{4} + 1}$$

for $k_2 \to -\infty$, $\omega_2 \to 0$ Rossby ! (westward)

$k_2 \to +\infty$, $\omega_2 \to k_2$ Kelvin ! (eastward)

Group velocity of Yanai waves:

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left[\frac{d}{d\omega}\left(\omega - \frac{1}{\omega}\right)\right]^{-1} = \left[1 + \frac{1}{\omega^2}\right]^{-1}$$

i.e. $\boxed{v_g = \frac{\omega^2}{1+\omega^2}}$ is always > 0

Note: Kelvin waves happen to be formally included as $m = -1$.

Why do they occur here?

Vallis: above analysis includes (by „accident") equatorially trapped = Kelvin waves propagating eastward only.

- Beautiful numerical confirmation of the Yanai wave on a full sphere: Vallis p. 312
- Beautiful observations of Yanai wave & the others: Vallis p. 718 for cloud brightness

Heuristic derivation   Vallis p. 314   (Sonntag 3.4.22

We had derived above, from linear combination of 4 equations for u, v, η, ζ, für $\beta = c = 1$

$$\boxed{f'' + \left(\omega^2 - k^2 - \frac{k}{\omega} - y^2\right) f = 0} \qquad f = f(y) \quad (*)$$

where $v(x,y,t) = f(y)\, e^{i(kx - \omega t)}$.

This is now derived – approximately – more directly

Start with vorticity equation, linearized

$$\frac{\partial}{\partial t}\left(\zeta - \frac{\beta y}{H}\eta\right) + \beta v = 0 \qquad \text{(this was above eq. 4)}$$

Assume near-geostrophic flow,

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \quad \zeta = \Delta\psi, \quad \eta = \frac{f}{g}\psi = \frac{\beta y}{g}\psi$$

Coriolis, not $f(y)$

gibt $\frac{\partial}{\partial t}\left(\Delta\psi - \frac{\beta^2 y^2}{c^2}\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0 \qquad (c^2 = gH)$

let $\psi(x,y,t) = \phi(y)\, e^{i(kx - \omega t)}$

dann

$$-i\omega\left[\phi'' - k^2\phi - \frac{\beta^2 y^2}{c^2}\phi\right] + ik\beta\phi = 0$$

d.h.

$$\boxed{\phi'' - \left[k^2 + \frac{\beta k}{\omega} + \frac{\beta^2 y^2}{c^2}\right]\phi = 0}$$

which is almost $(*)$: 1) ω^2 is missing since we are in the low-frequency limit. 2) And $f \sim v = \psi_x \sim k\phi$ thus f and ϕ obey the same equation.

The same after Pedlosky: p. 586!

Equatorial waves = Poincaré, Kelvin, Rossby

↑ This was missing so far, but then we assumed

$\boxed{f = \beta y}$ without f_0

Since $f \to 0$ at equator, no geostrophic balance

→ Huge chapter on equatorial dynamics

Here only equatorially trapped (Kelvin) waves

Poincaré: $\omega_p^2 = f^2 + g\overset{\text{" const}}{H}(k^2+l^2)$

Rossby: $\omega_R = -\dfrac{\beta k}{k^2 + l^2 + f^2/gH}$

$\min(\omega_p) = f$, and $\omega_R \to 0$

Thus for $f \to 0$, Poincaré & Rossby have no gap and may develop common mode = Yanai wave

$$\omega_p = f^2 + gH(k^2 + l^2)$$

for fixed ω_p and k, l^2 must become < 0 with growing f^2: evanescent wave in y-direct.

→ Equatorial region acts as wave guide

In a band about the equator $\boxed{f = \beta y}$

with $\beta = 2\Omega/R_E$
radius of Earth

equations of motion, as before, normalized, with $\beta \equiv 1$,

$$\frac{\partial u}{\partial t} - yv = \left(-\frac{\partial p}{\partial x}\right) \equiv -\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + yu = \left(-\frac{\partial p}{\partial y}\right) \equiv -\frac{\partial \eta}{\partial y} \quad (2)$$ 3 Gleichungen

$$\boxed{m^2 \frac{\partial \eta}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0} \quad (3)$$ für u, v, η

Nach Moore & Philander (1977) wurde

$\dot\eta + H\,\nabla\cdot\vec u$ ersetzt durch $m^2\dot\eta + \nabla\cdot\vec u$

Für (1),(2),(3) siehe Pedlosky S. 590 (8.5.19a, b), (8.5.20) (mit $p \equiv \eta$)

Eliminiere η mit Hilfe von (3) in (1), (2):

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 u}{\partial x^2} = y\frac{\partial v}{\partial t} + \frac{1}{m^2}\frac{\partial^2 v}{\partial x \partial y} \quad (1')$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 v}{\partial y^2} = -y\frac{\partial u}{\partial t} + \frac{1}{m^2}\frac{\partial^2 u}{\partial x \partial y} \quad (2')$$

Eliminiere u. Wende $\partial_t^2 - \frac{1}{m^2}\partial_x^2$ auf (2') an:

$$\left(\partial_t^2 - \frac{1}{m^2}\partial_x^2\right)\left(\partial_t^2 - \frac{1}{m^2}\partial_y^2\right)v \doteq \left(\partial_t^2 - \frac{1}{m^2}\partial_x^2\right)\left(-y\partial_t + \frac{1}{m^2}\partial_x\partial_y\right)u$$

$$= \left(-y\partial_t + \frac{1}{m^2}\partial_x\partial_y\right)\left(\partial_t^2 - \frac{1}{m^2}\partial_x^2\right)u$$

$$\overset{(1')}{=} \left(-y\partial_t + \frac{1}{m^2}\partial_x\partial_y\right)\left(y\partial_t + \frac{1}{m^2}\partial_x\partial_y\right)v$$

don't forget!

$$\left(\partial_t^4 - \frac{1}{m^2}\partial_t^2(\partial_x^2 + \partial_y^2) + \frac{1}{m^4}\partial_x^2\partial_y^2\right)v = \left(\frac{1}{m^4}\partial_x^2\partial_y^2 - y^2\partial_t^2 + \frac{1}{m^2}\partial_x\frac{\partial y}{\partial y}\partial_t\right)v$$

$$\left(m^2\partial_t^4 - \partial_t^2(\partial_x^2 + \partial_y^2) + m^2 y^2\partial_t^2 - \partial_t\partial_x\right)v = 0$$

Ein ∂_t kann weg (Konstante 0!)

$$\boxed{\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v - m^2y^2 v - m^2\frac{\partial^2 v}{\partial t^2}\right] + \frac{\partial v}{\partial x} = 0}$$

also deutlich übersichtlicher als vorherige Ableitung in Vallis mit Lin. komb. von 4 Gleichungen (potential vorticity war überflüssige Glg!):

$$-\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v + \frac{f^2}{c^2}v + \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}\right] - \beta\frac{\partial v}{\partial x} = 0$$

in Vallis. Beachte Übereinstimmung der Glgen für $\beta \equiv 1$ und $f = \beta y$ und $m^2 = 1/c^2$

Obige Gleichung im Kasten: Pedlosky S. 590 (8.5.23)

Nebenbetrachtung: es kann u-Wellen mit $v = 0$ geben

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

$$\frac{\partial^2 u}{\partial x \partial y} - m^2 y\frac{\partial u}{\partial t} = 0 \qquad (2)$$

aus (1): $u = U_\pm\left(x \pm \frac{t}{m}, y\right)$.

in (2): $\partial_y \partial_x U_\pm \mp m y\, \partial_x U_\pm = 0$

d.h. $\boxed{U_\pm = h_\pm\left(x \pm \frac{t}{m}\right) e^{\pm\frac{1}{2} m y^2}}$

erstmals hier der Faktor $e^{-my^2/2}$! wähle $m > 0$

$$\boxed{U = h\left(x - \frac{t}{m}\right) e^{-\frac{1}{2} m y^2}} \qquad (8.5.28)$$

nach Osten laufende Kelvinwelle am Äquator (equatorially trapped wave)

← Nun zurück hierhin. Ansatz

$$v(x, y, t) = \Psi(y)\, e^{i(kx - \omega t)}$$

gilt $\boxed{\frac{d^2\Psi}{dy^2} + \left(m^2(\omega^2 - y^2) - \frac{k}{\omega} - k^2\right)\Psi = 0}$ ⊛

wie zuvor (für f)

Lösungen sind wieder

$$\Psi_j(y) = \text{const}_j\, e^{-\frac{1}{2}\tilde{y}^2} H_j(\tilde{y}) \quad \text{mit } \tilde{y} = y\sqrt{m}$$

mit Hermitepolynom definiert durch

$$\boxed{H_j(y) = (-1)^j e^{y^2} \frac{d^j}{dy^j} e^{-y^2}}$$

Setzt man dies in die DGL oben ein, so erhält man

Ü $\boxed{m^2\omega^2 - \frac{k}{\omega} - k^2 = (2j+1)m}$ $(j = 0, 1, 2, \ldots)$

nämlich die Ψ_j erfüllen die DGL

$$\frac{d^2\Psi_j}{d\tilde{y}^2} + \left((2j+1) - \tilde{y}^2\right)\Psi_j = 0$$

also in y:

$$\frac{1}{m}\frac{d^2\Psi_j}{dy^2} + \left((2j+1) - m y^2\right)\Psi_j = 0$$

oder $\frac{d^2\Psi_j}{dy^2} + \left((2j+1)m - m^2 y^2\right)\Psi_j = 0$

was direkt mit ⊛ verglichen werden kann

Setze $j = 0$: $m^2 \omega^2 - \frac{k}{\omega} - k^2 = m$

hat Lösungen $k = -m\omega$: $m^2\omega^2 + m\omega - m^2\omega^2 = m\omega$ ✓

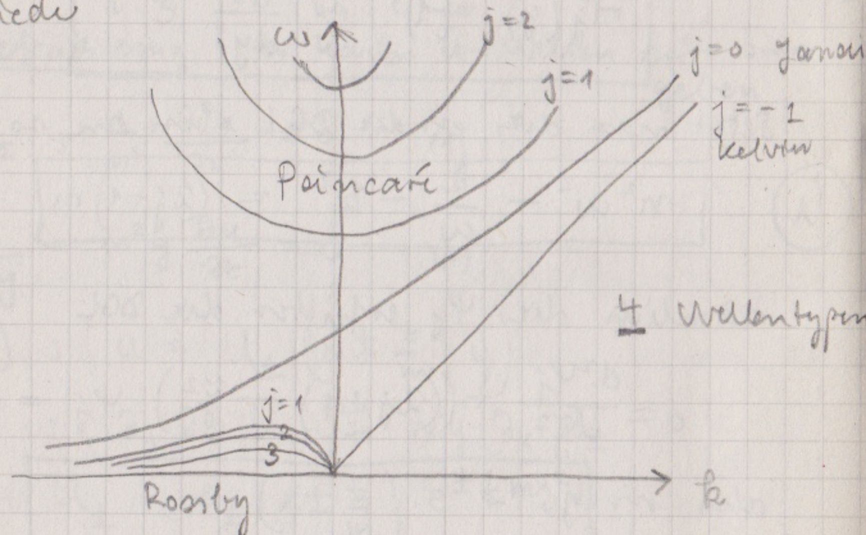
und $\boxed{k = -\frac{1}{\omega} + m\omega}$: $\cancel{m^2\omega^2} + \cancel{\frac{1}{\omega^2}} - m - \cancel{\frac{1}{\omega^2}} - \cancel{m^2\omega^2} + 2m\omega = m$

Hier: $\frac{\omega}{k} = -\frac{1}{m}$ westwärts laufende Kelvinwelle

und $k = -\frac{1}{\omega} + m\omega$ ist Yanai welle;

für $\begin{cases} \omega \to \infty : & \frac{\omega}{k} = +\frac{1}{m} \quad \text{ostwärts laufende Kelvinw} \\ \omega \to 0 : & \frac{\omega}{k} = -\frac{1}{k^2} \quad \text{westwärts laufende Rossby} \end{cases}$

Also wieder



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